In the previous lecture, we learned how to solve

\[ ay'' + by' + cy = 0, \]

using three steps:

1. Guess the form of the solution \( e^{\lambda x} \) to determine the characteristic equation: \( a\lambda^2 + b\lambda + c = 0 \)
2. Solve the characteristic equation to find the fundamental solutions: \( y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x} \)
3. Combine the fundamental solutions (via superposition theorem) to determine the general solution:
   \[ y = c_1 y_1 + c_2 y_2 \]

It turns out that we can use this process to solve any linear constant-coefficient ODE of any order! However, so far, we have restricted ourselves to constant-coefficient 2nd order ODE whose characteristic equation does not have repeated roots.

If we slightly generalize the class of ODEs that we are considering to any linear second order ODE of the form

\[ y'' + p(x)y' + q(x)y = 0, \]

the superposition theorem will still hold. Thus, the general solution is given by \( y = c_1 y_1 + c_2 y_2 \), where \( y_1 \) and \( y_2 \) are two linearly independent solutions of

\[ y'' + p(x)y' + q(x)y = 0. \]

But before we talk about linear independence of fundamental solutions, perhaps a more basic question we should ask is the following:

If \( y'' + p(x)y' + q(x)y = 0 \), and we’re given initial conditions \( y(x_0) = y_0 \) and \( y'(x_0) = y'_0 \), how do we even know if a solution will exist?

In this lecture, we will talk about the existence of solutions, the concept of linear independence, and the Wronskian.
EXISTENCE, UNIQUENESS, AND THE WRONSKIAN

**Theorem**

Consider the initial value problem

\[ y'' + p(x)y' + q(x)y = g(x), \quad y(x_0) = y_0, y'(x_0) = y'_0, \]

where \( p(x), q(x) \) and \( g(x) \) are continuous on an open interval \( I \) that also contains \( x_0^1 \). Then, there exists exactly one solution \( y = \phi(x) \) of this problem and the solution exists throughout the interval \( I \).

It’s important to note the three things that this theorem says:

1. The initial value problem has a solution; in other words, a solution exists.
2. The initial value problem has only one solution; in other words, the solution is unique.
3. The solution \( \phi(x) \) is defined throughout the interval \( I \) where the coefficients are continuous and is at least twice differentiable there.

**Example**

Find the longest interval in which the solution of the initial value problem

\[ (t^2 - 3t) \frac{d^2 y}{dt^2} + t \frac{dy}{dt} - (t + 3)y = 0, \quad y(1) = 2, \quad \left. \frac{dy}{dt} \right|_{t=1} = 1 \]

**Solution**

In this problem, if we write it in the form where the coefficient of the second derivative term is one, we find that \( p(t) = 1/(t - 3), \) \( q(t) = -(t + 3)/(t^2 - 3t), \) and \( g(t) = 0. \) The only points of discontinuity of the coefficients are at \( t = 0 \) and \( t = 3. \) Therefore, the longest open interval containing the initial point \( t = 1 \) in which all of the coefficients are continuous is \( I = 0 < t < 3. \) Thus, this is the longest interval in which our theorem guarantees that a solution exists.

**LINEAR INDEPENDENCE AND THE WRONSKIAN**

Now, let’s return to a question we posed earlier:

Assume that both \( y_1 \) and \( y_2 \) are two solutions of

\[ y'' + p(x)y' + q(x)y = 0. \]

Superpositions tells me that \( y = c_1 y_1 + c_2 y_2 \) is the general solution of the ODE if \( y_1 \) and \( y_2 \) are linearly independent. But how can we determine if two solutions are linearly independent?
**Definition: Linear Independence**

Two functions $f(x)$ and $g(x)$ are called *linearly independent* if the equation

$$c_1 f(x) + c_2 g(x) = 0, \text{ for all } x,$$

can only be satisfied by choosing $c_1 = 0$ and $c_2 = 0$. Two functions that are not linearly independent are called *linearly dependent*.

This is very similar to the concept for linearly independent vectors. For example, consider the two vectors $v_1$ and $v_2$. These vectors are linearly independent if $c_1 v_1 + c_2 v_2 = 0$ implies that both $c_1$ and $c_2$ are zero. If you need more verification, consider the concrete example where $v_1 = [10]^T$ and $v_2 = [01]^T$. It should be clear that they are linearly independent vectors. Furthermore, the only way that we can get the zero vector if we take a linear combination is by letting both $c_1$ and $c_2$ be identically zero.

**Example**

Determine if $f(x) = e^x$ and $g(x) = 2e^x$ are linearly independent

**Solution**

$f(x)$ and $g(x)$ are linearly dependent because

$$-2f(x) + g(x) = 0,$$

so $c_1 = -2$ and $c_2 = 1$. If the only choice was to choose them both zero, the functions would be independent.

Wouldn’t it be nice if there was an easier way to determine linear independence? Well, there is! We need to introduce the Wronskian first.

**Definition: The Wronskian**

The *Wronskian* of two function $f(x)$ and $g(x)$ is just the quantity

$$W(f, g)(x) = f(x)g'(x) - f'(x)g(x).$$

It’s sometimes easier to think of the Wronskian using matrix notations. In other words:

$$W(f, g)(x) = \det \begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} = f(x)g'(x) - f'(x)g(x).$$
Example

Let \( f(x) = \sin(x) \) and \( g(x) = \cos(x) \), find \( W(f,g)(x) \).

Solution

\[
W(f,g)(x) = \sin(x)(-\sin(x)) - \cos(x)(\cos(x)) = -1.
\]

Theorem

Two functions \( f(x) \) and \( g(x) \) are linearly dependent if their Wronskian

\[
W(f,g)(x) = f(x)g'(x) - f'(x)g(x) = 0.
\]

Proof

If \( f(x) \) and \( g(x) \) are linearly dependent, then we can find constants \( c_1 \) and \( c_2 \), not both zero, so that

\[
c_1 f(x) + c_2 g(x) = 0, \text{ for all } x.
\]

Then also

\[
c_1 f'(x) + c_2 g'(x) = 0, \text{ for all } x.
\]

Now, let’s assume that \( f \neq 0 \), otherwise we can switch the roles of \( f \) and \( g \). Then we can solve to find

\[
c_1 = -\frac{c_2 g(x)}{f(x)}
\]

\[
c_1 f'(x) + c_2 g'(x) = -\frac{c_2 g(x)}{f(x)} f'(x) + c_2 g'(x)
\]

\[
= \frac{c_2}{f(x)} (g(x)f'(x) - g'(x)f(x)) = 0
\]

Note that \( c_2 \neq 0 \), since otherwise \( c_1 \) would also be zero, which would imply the functions are linearly independent. Thus plugging the expression into the equation for the derivative, we have

\[
f g' - f' g = 0 \implies W(f,g)(x) = 0,
\]

which is what we had to prove.
**Example**
Calculate the Wronskian for the functions $f(x) = e^x$ and $g(x) = 2e^x$ to determine if they are linearly independent.

**Solution**

$$W(e^x, 2e^x) = e^x(2e^x) - e^x(2e^x) = 0,$$

Since the Wronskian is equal to zero, the two functions are linearly dependent.

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**Example**
Calculate the Wronskian for the functions $f(x) = e^x$ and $g(x) = e^{-x}$ to determine if they are linearly independent.

**Solution**

$$W(e^x, e^{-x}) = e^x(e^{-x}) - e^x(-1e^{-x}) = 1 - (-1) = 2,$$

Since the Wronskian is not equal to zero, the two functions are linearly independent.