In this lecture, we’ll look at second-order equations for the first time. All the second-order equations we’ll consider here will be linear. We won’t look at nonlinear equations again until we get to nonlinear systems, much later in these notes.

Any second-order linear equation is of the form
\[ r(x)y'' + p(x)y' + q(x)y = g(x), \]
where \( r(x) \), \( p(x) \) and \( q(x) \) may be functions of \( x \). For now, we’ll assume they are constants: \( r(x) = a \), \( p(x) = b \) and \( q(x) = c \), with \( a \), \( b \) and \( c \) constant. Further, we’ll start with the \textbf{homogeneous case}, \textit{i.e.}, the case where \( g(x) = 0 \). Thus, we’ll consider
\[ ay'' + by' + cy = 0, \]
where \( y = y(x) \) is the function we’re looking for. Let’s introduce some shorthand. Let
\[ L[y] = ay'' + by' + cy, \]
so that the differential equation simply is \( L[y] = 0 \). Let’s discuss a few properties of this equation.

**Theorem: Principle of Superposition**

*If \( y_1 \) and \( y_2 \) are independent\(^*\) solutions of this equation, then \( y(x) = c_1 y_1(x) + c_2 y_2(x) \) is the general solution.

\(^*\)We will precisely define independence later on in the lecture.

**Proof**

\[ L[y] = ay'' + by' + cy \]
\[ = a(c_1y_1'' + c_2y_2'') + b(c_1y_1' + c_2y_2') + (c_1y_1 + c_2y_2) \]
\[ = c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2) \]
\[ = c_1L[y_1] + c_2L[y_2] \]
\[ = c_10 + c_20 \]
\[ = 0. \]

We’ve used that \( L[y_1] = 0 \) and \( L[y_2] = 0 \), since \( y_1 \) and \( y_2 \) are solutions. Thus \( y_1 = c_1 y_1 + c_2 y_2 \) is also a solution, which is what we had to prove.
We can use this theorem to get new solutions from known ones: if $y_1$ and $y_2$ are solutions, then so are $y_3 = (y_1 + y_2)/2$ and $y_2 = (y_1 - y_2)/2$. These are easily obtained by choosing $c_1 = c_2 = 1/2$, and $c_1 = 1/2$, $c_2 = -1/2$ in the theorem.

In order for the theorem to hold, $y_1$ and $y_2$ have to be “independent”. What does this mean? We’ll define this properly soon, but for the moment it suffices to say that $y_1$ and $y_2$ are not a multiple of each other. If this happens, say $y_2 = \alpha y_1$, for some constant $\alpha$, then

$$
y = c_1y_1 + c_2y_2 = c_1y_1 + c_2\alpha y_1 = (c_1 + c_2\alpha)y_1 = c_3y_1,
$$

where $c_3 = c_1 + c_2\alpha$ is another constant. We see that in this case, our proposed general solution $y$ only depends on one constant. That’s not enough!

**Here’s why this theorem absolutely rocks:** in order to find the general solution of

$$
L[y] = 0,
$$

it suffices to find two solutions $y_1$ and $y_2$! Awesome!

It’s easy to find two such solutions: guess

$$
y = e^{\lambda x},
$$

for some constant $\lambda$, to be determined. Then

$$
y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}.
$$

Plugging all this in, we get

$$
a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0
$$

$$
e^{\lambda x} (a\lambda^2 + b\lambda + c) = 0
$$

$$
a\lambda^2 + b\lambda + c = 0,
$$

since $e^{\lambda x}$ is never zero. Thus, in order to find solutions, we have to choose $\lambda$ to be a solution of the quadratic equation

$$
a\lambda^2 + b\lambda + c = 0.
$$

This equation is known as the **Characteristic equation** of the differential equation. From it, we get two solutions for $\lambda$:

$$
\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
$$

This gives two solutions of the original differential equation, namely

$$
y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}.
$$

Using our theorem, we find that the general solution is

$$
y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.
$$
Thus, we’ve constructed the general solution for a second-order linear equations with constant coefficients, and all we’ve had to do was solve a quadratic equation!

This works very well if $\lambda_1$ and $\lambda_2$ are both real, and different. In the other cases, we’ll have to do a bit of extra work. Let’s look at some examples where the above does work.

**Example**

Find the solution to the IVP:

$$y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1,$$

**Solution**

Note that we’re specifying two initial conditions, since we have two constants to determine. Let’s start with the assumption that the solutions take the form

$$y = e^{\lambda x},$$

where $\lambda$ is an unknown constant. Since we made an assumption for the solution, we better check to see if it actually solves the ODE. In order to plug in $y = e^{\lambda x}$, we need to calculate $y'$ and $y''$. This gives the following information that we should plug into the ODE:

$$y = e^{\lambda x}, \quad y' = \lambda e^{\lambda x}, \quad 	ext{and} \quad y'' = \lambda^2 e^{\lambda x}.$$

Plugging this into the ODE $y'' - y = 0$, we find

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \quad \Rightarrow \quad \lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = -1,$$

from which $y_1 = e^x$, $y_2 = e^{-x}$, and the general solution is

$$y = c_1 e^x + c_2 e^{-x}.$$

Since we’ll need $y'$ to use the second initial condition, let’s calculate it now: $y' = c_1 e^x - c_2 e^{-x}$. Plugging in the two initial conditions, we get

$$y(0) = c_1 + c_2, \quad y'(0) = c_1 - c_2,$$

so that $c_1 + c_2 = 2$ and $c_1 - c_2 = -1$. Adding and subtracting these two equations we find that $c_1 = 1/2$ and $c_2 = 3/2$. Finally, the solution of the initial-value problem is

$$y = \frac{1}{2} e^x + \frac{3}{2} e^{-x}. $$
Example

Find the general solution to the ode:

\[ y'' + 5y' + 6y = 0. \]

Solution

The characteristic equation is

\[ \lambda^2 + 5\lambda + 6 = 0 \]

\[ \Rightarrow \]

\[ (\lambda + 2)(\lambda + 3) = 0 \]

\[ \Rightarrow \]

\[ \lambda_1 = -2, \quad \lambda_2 = -3, \]

and thus \( y_1 = e^{-2x}; \) \( y_2 = e^{-3x}. \) The general solution is

\[ y = c_1y_1 + c_2y_2 = c_1e^{-2x} + c_2e^{-3x}. \]