MATH 361
Homework #3
Winter Quarter 2014
Due: February 5, 2014 (by 5pm)

Total Possible Points: 60

1. [Warm-up for Question #2]. Consider the first-order differential equation

\[ y' = y(y - a) \]  \hspace{1cm} (1)

where \( a \) is some arbitrary constant parameter. Let \( f(y; a) \) represent the function on the right-hand side of the differential equation given by \( f(y; a) = y(y - a) \).

(a) (3 points) Draw the phase line for the Equation (1) corresponding to \( a^* = 1 \). Discuss the stability of all equilibrium points in this scenario.

(b) (3 points) What would you expect to happen to the stability of the equilibrium points from part (a) if you change the value of the parameter \( a \) “just a little bit”? How big do you think you can make “just a little bit” before things change? Write 2-3 sentences explaining your logic.

(c) (3 points) Now, we will formally justify what you expect to be true. Show that \((y^*, a^*) = (1, 1)\) satisfies the conditions

\[ f(y^*; a^*) = 0, \quad \text{and} \quad \left. \frac{\partial f}{\partial y} \right|_{(y^*, a^*)} \neq 0. \]

(d) (6 points) Now, consider the differential equation given by

\[ y' = f(y; a^* + \epsilon) = y(y - (a^* + \epsilon)) \]  \hspace{1cm} (2)

where \( a^* = 1 \). Prove that the new differential equation (2) has an equilibrium point \( y^*(\epsilon) \) where the map from \( \epsilon \to y^*(\epsilon) \) is a smooth function satisfying \( y^*(0) = 1 \) (the equilibrium point from parts (a) and (b)) for \( \epsilon \) sufficiently small.

(e) (3 points) From the previous part, how large can you make \( \epsilon \) before things break down? Can you relate this to part (b)?

2. (15 points) [Chapter 9, Problem 3 from Hirsch, Smale, and Devaney]. Consider the first-order differential equation

\[ y' = f(y; a), \]

where \( a \) is some parameter in the problem. Now, let \((y^*, a^*)\) represent a constant \( y \) and \( a \) value such that

\[ f(y^*; a^*) = 0, \quad \text{and} \quad \left. \frac{\partial f}{\partial y} \right|_{(y^*, a^*)} \neq 0. \]

Prove that the differential equation

\[ y' = f(y^*; a^* + \epsilon), \]

has an equilibrium point \( y^*(\epsilon) \) where the map from \( \epsilon \to y^*(\epsilon) \) is a smooth function satisfying \( y^*(0) = y^* \) for \( \epsilon \) sufficiently small.
3. Consider two species of giant squid: the orange giant squid and the red giant squid in an area of the ocean. Let \( x(t) \) represent the population size of the orange giant squid and \( y(t) \) represent the population size of the red giant squid. In this system, consider the following:

- The resources for both varieties of squid are limited.
- Both species of squid compete with each other for the same food source.
- There is a migration of species \( x \) (that is, from somewhere else, species \( x \) increases at a constant rate).

(a) (5 points) Formulate one system of differential equations describing all of the following interactions for 2 populations \( x(t) \) and \( y(t) \).

Solution:

\[
\frac{dx}{dt} = rx - bx^2 - cxy + f \quad \frac{dy}{dt} = ky - ay^2 - dxy
\]

(b) (8 points) Find the non-dimensional form of the equations. You should have only 4 parameters remaining (one of which corresponds to the migration rate).

Solution: For this, let \( x = \alpha M \), \( y = \beta N \) and \( t = \gamma \tau \). Plugging these into the differential equation gives:

\[
\frac{\alpha dM}{\gamma d\tau} = \alpha r M - b\alpha^2 M^2 - c\alpha \beta \gamma N + f \\
\frac{dM}{d\tau} = r\gamma M \left( 1 - \frac{b\alpha}{r} M - \frac{c\beta}{r} N \right) + \frac{\gamma}{\alpha} f
\]

Likewise, for the second equation \( \frac{dy}{d\tau} \), we get

\[
\frac{\alpha dN}{\gamma d\tau} = \beta k N - a\beta^2 N^2 - d\alpha \beta M N \\
\frac{dN}{d\tau} = k\gamma N \left( 1 - \frac{a\beta}{k} N - \frac{d\alpha}{k} M \right)
\]

Looking at the equation for \( \frac{dM}{d\tau} \), we see that if we let \( \alpha = r/b \), we can get rid on one parameter. Looking at the second equation, we have a similar thing that we can do. If we let \( \beta = k/a \), we also have eliminated a parameter.

So far, we have reduced our equations to

\[
\frac{dM}{d\tau} = r\gamma M \left( 1 - M - \frac{ck}{ar} N \right) + \frac{b\gamma}{r} f \\
\frac{dN}{d\tau} = k\gamma N \left( 1 - N - \frac{dr}{kb} M \right)
\]

There is one last parameter to choose. We have yet to pick an appropriate value for \( \gamma \). There are actually two obvious choices that we can make; (1) \( \gamma = 1/r \) or (2) \( \gamma = 1/k \). Let’s just choose the first option. Plugging this into the ODE, we have

\[
\frac{dM}{d\tau} = M (1 - M - nN) + p \\
\frac{dN}{d\tau} = \delta N (1 - N - mM)
\]

where \( n = \frac{ck}{ar} \), \( m = \frac{dr}{kb} \), \( \delta = \frac{k}{r} \), and \( p = \frac{bf}{r^2} \).
(c) (8 points) Find the equilibrium points for the nondimensional system. Compare the location of the equilibrium points with those of the nondimensional competition model talked about in class.

**Solution:** There are a lot of equilibrium points that we have to consider. Let’s begin with the second equation.

\[ 0 = N(1 - N - mM) \]

From this, we see that \( N^* = 0 \) or \( N^* = 1 - mM^* \). We can use both of these relationships in the first equation. If \( N^* = 0 \)

\[ 0 = M(1 - M) + p \quad \rightarrow \quad M^2 - M - p = 0 \]

Solving for \( M \), we have \( M^* = \frac{1}{2} \left( 1 \pm \sqrt{1 + 4p} \right) \). This means, so far, we have two equilibrium points:

\[ p_1 = \left( \frac{1}{2} \left( 1 + \sqrt{1 + 4p} \right), 0 \right) \quad p_2 = \left( \frac{1}{2} \left( 1 - \sqrt{1 + 4p} \right), 0 \right) \]

Now if we let \( N^* = 1 - mM^* \) and plug this into the first equation we have

\[ 0 = M(1 - M - n(1 - mM)) + p \quad \rightarrow \quad (1 - nm)M^2 - (1 - n)M - p = 0 \]

Solving for \( M \), we have \( M^* = \frac{1}{2} \left( 1 - nm \right) \left( (1 - n) \pm \sqrt{(1 - n)^2 + 4p} \right) \). This means that there are two additional equilibrium points:

\[ p_3 = \left( \frac{1}{2} \left( 1 - nm \right) \left( (1 - n) + \sqrt{(1 - n)^2 + 4p} \right), 1 - \frac{m}{2(1 - nm)} \left( (1 - n) + \sqrt{(1 - n)^2 + 4p} \right) \right) \]

\[ p_4 = \left( \frac{1}{2} \left( 1 - nm \right) \left( (1 - n) - \sqrt{(1 - n)^2 + 4p} \right), 1 - \frac{m}{2(1 - nm)} \left( (1 - n) - \sqrt{(1 - n)^2 + 4p} \right) \right) \]

(d) **Bonus: (5 points)** Investigate the effect of changing the migration rate if both species are not strong competitors. You may do this using any computer software that you have available.

4. (6 points) Two species, \( A \) and \( B \) are in competition and are the prey of a third species \( C \). Write down a system of differential equations that describe this ecological system.

**Solution:**

\[
\begin{align*}
\frac{dA}{dt} &= aA - bA^2 - cAB - dAC \\
\frac{dB}{dt} &= eA - fA^2 - gAB - hBC \\
\frac{dC}{dt} &= jAC - kBC - mC
\end{align*}
\]