Homework 7 (due November 12, 2009)

Problem 1. Show that
\[ E[Y] = \int_0^\infty P\{Y > y\}dy - \int_0^\infty P\{Y < -y\}dy. \]

First show the following two equalities:
\[ \int_0^\infty P\{Y > y\}dy = \int_0^\infty \int_y^\infty f_Y(x)dxdy = \int_0^\infty \int_0^x f_Y(x)dydx = \int_0^\infty xf_Y(x)dx \]
\[ \int_0^\infty P\{Y < -y\}dy = \int_0^\infty \int_{-\infty}^{-y} f_Y(x)dxdy = \int_{-\infty}^0 \int_0^{-x} f_Y(x)dydx = \int_0^0 -xf_Y(x)dx \]

Here we have used the definition of the continuous random variable \( Y \), and some changes in orders of integration. We combine these two facts with the original statement we wish to prove, equating the right hand side of the equation gives:
\[ \int_0^\infty P\{Y > y\}dy - \int_0^\infty P\{Y < -y\}dy = \int_0^\infty xf_Y(x)dx + \int_{-\infty}^0 xf_Y(x)dx = \int_{-\infty}^\infty xf_Y(x)dx = E[Y]. \]

Problem 2. Let \( X \) be a continuous random variable. Using the definition of expectation value for continuous random variables verify that:
\[ (a) \quad E[aX + b] = aE[X] + b \quad \text{for constants} \ a, b \]
Solution: This is a straight forward computation that goes as follows:

\[ E[aX + b] = \int_{-\infty}^{\infty} (ax + b) f_X(x) dx = a \int_{-\infty}^{\infty} x f_X(x) dx + b \int_{-\infty}^{\infty} f_X(x) dx = aE[X] + b. \]

(b) \[ Var(X) = E[X^2] - (E[X])^2 \]

Solution: This is a similar argument as above.

Problem 3. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M.

(a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A?

Solution: If we let \( X \) equal the arrival time in minutes after 7 which the passenger arrives, then \( X \) is a uniform random variable on \((0, 60)\). The times which the passenger gets on the train to go to destination A are the time intervals when \(15K - 10 < X \leq 15K\) for \( K = 1, 2, 3, 4 \). Thus the probability that the passenger arrives in those times if given by:

\[ \sum_{k=1}^{4} \int_{15K-10}^{15K} \frac{1}{60} dx = \sum_{k=1}^{4} \frac{1}{6} = \frac{2}{3}. \]

(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 A.M.?

Solution: In the same manner as above we let \( X \) be the time in minutes after 7:10 that the passenger arrives is equal to:

\[ P\{0 < X \leq 5\} + \sum_{k=1}^{3} P\{15K - 5 < X \leq 15K + 5\} + P\{55 < X \leq 60\} \]

\[ = \int_{0}^{5} \frac{1}{60} dx + \sum_{k=1}^{3} \int_{15K-5}^{15K+5} \frac{1}{60} dx + \int_{55}^{60} \frac{1}{60} dx \]

\[ = \frac{1}{12} + 3 \frac{1}{6} + \frac{1}{12} = \frac{2}{3}. \]

Problem 4. A point is chosen at random on a line segment of length L. Interpret this
statement and find the probability that the ratio of the shorter to the longer segment is less then \( \frac{1}{4} \).

**Solution:** Assume that \( X \) is a random variable that is equal to the distance from the right hand hand of the line. Then if we assume that \( X \) is uniformly distributed over \((0, L)\). The probability that we are interested in is that:

\[
P\left\{ \min \left( \frac{X}{L-X}, \frac{L-X}{X} \right) < \frac{1}{4} \right\} \]

\[
= 1 - P\left\{ \min \left( \frac{X}{L-X}, \frac{L-X}{X} \right) \geq \frac{1}{4} \right\} 
\]

\[
= 1 - P\left\{ \frac{X}{L-X} \geq \frac{1}{4} \text{ and } \frac{L-X}{X} \geq \frac{1}{4} \right\} 
\]

\[
= 1 - P\left\{ X \geq \frac{L}{5}, \text{ and } X \leq \frac{4L}{5} \right\} 
\]

\[
= 1 - \int_{\frac{L}{5}}^{\frac{4L}{5}} \frac{1}{L} \, dx = 1 - \frac{3}{5} = \frac{2}{5}. 
\]

**Problem 5.** The speed of a molecule in a uniform gas at equilibrium is a random variable whose probability density function is given by

\[
f(x) = \begin{cases} 
    ax^2e^{-bx^2} & x \geq 0 \\
    0 & x < 0 
\end{cases}
\]

where \( b = \frac{m}{2kT} \) and \( k, T, \) and \( m \) denote, respectively, Boltzmann’s constant, the absolute temperature, and the mass of the molecule. Evaluate \( a \) in terms of \( b \). And compute the expected value of \( X \) in terms of \( b \).
**Solution:** To find $a$ in terms of $b$ we integrate over $0$ to $\infty$ and set the integral equal to 1:

$$1 = a \int_{0}^{\infty} x^2 e^{-bx^2} \, dx$$

Now we integrate by parts letting $u = x$ and $dv = xe^{-bx^2} \, dx$, which gives:

$$1 = a \left\{ \left. -\frac{x}{2b} e^{-bx^2} \right|_{0}^{\infty} + \frac{1}{2b} \int_{0}^{\infty} e^{-bx^2} \, dx \right\}$$

$$= a \frac{\sqrt{\pi}}{4b^{\frac{3}{2}}}.$$ 

Thus $a = \frac{4b^{\frac{3}{2}}}{\sqrt{\pi}}$. Computing the expectation value is a similar computation using change of variables that would give $E[X] = \frac{2}{\sqrt{b\pi}}$. (An answer is not enough, need to show how you did the integration)

**Problem 6.** Let $X$ be a random variable that takes on values between 0 and $c$. That is $P\{0 \leq X \leq c\} = 1$. Show that

$$Var(X) \leq \frac{c^2}{4}.$$ 

**Solution:** First consider that if $x \in (0, c)$ then $x \leq c$ and thus we have that since $x$ and $f(x)$ are both strictly non-negative in this case, then $x^2 f(x) \leq cx f(x)$. Integrating both sides of this expression gives that:

$$\int_{0}^{c} x^2 f(x) \, dx \leq c \int_{0}^{c} x f(x) \, dx,$$

or $E[X^2] \leq cE[X]$. Now we can write the variance as:

$$Var(X) = E[X^2] - (E[X])^2 \leq cE[X] - E[X]^2.$$ 

If we let $\alpha = \frac{E[X]}{c}$ then we have that:

$$Var(X) \leq c^2 \alpha - c^2 \alpha^2 = c^2 \alpha (1 - \alpha).$$

If we consider the function $g(\alpha) = \alpha(1 - \alpha)$ and recognize that this is a quadratic function of $\alpha$ which has a concave-down parabola as its graph then it has a maximum value that occurs when the derivative is 0. This occurs when $\alpha = \frac{1}{2}$. Thus $f(\alpha) \leq f(\frac{1}{2}) = \frac{1}{4}$. Consequently, combining this with the above statement we get that:

$$Var(X) \leq c^2 \alpha (1 - \alpha) \leq c^2 \frac{1}{4}.$$
**Problem 7.** From a set of $n$ elements a nonempty subset is chosen at random in the sense that all of the nonempty subsets are equally likely to be selected. Let $X$ denote the number of elements in the chosen subset. Show that:

$$E[X] = \frac{n}{2 - \left(\frac{1}{2}\right)^{n-1}}$$

**Solution:** If we consider first compute $P\{X = i\}$ then we need to count all the subsets of size $i$. This is clearly counted by $\binom{n}{i}$ and since there are a total of $2^n - 1$ total possible subsets to choose from then we must have that: $P\{X = i\} = \frac{\binom{n}{i}}{2^n - 1}$. Consequently the expected value is given by:

$$E[X] = \frac{1}{2^n - 1} \sum_{k=1}^{n} k \binom{n}{k} = \frac{n}{2^n - 1} \sum_{k=1}^{n} \left(\frac{n-1}{k-1}\right)$$

Here we have used the identity that $k \binom{n}{k} = n \binom{n-1}{k-1}$. Now we reindex in the last sum letting $j = k - 1$ giving:

$$E[X] = \frac{n}{2^n - 1} \sum_{j=0}^{n-1} \left(\frac{n-1}{j}\right) = \frac{n2^{n-1}}{2^n - 1}$$

Where the last integral is a fact we verified in Chapter 2. A simple algebraic manipulation gives the equation in the form above.
\[ \text{Var}(X) = \frac{n \cdot 4^{n-1} - n(n+1)2^{n-2}}{(2^n - 1)^2} \]

**Solution:** This computation is similar to that above, using the reindexing trick twice:

\[
E[X^2] = \frac{1}{2^n - 1} \sum_{k=1}^{n} k^2 \binom{n}{k} \\
= \frac{n}{2^n - 1} \sum_{k=1}^{n} k \binom{n-1}{k-1} \\
= \frac{n}{2^n - 1} \sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} \\
= \frac{n}{2^n - 1} \sum_{j=1}^{n-1} j \binom{n-1}{j} + \frac{n}{2^n - 1} \sum_{j=0}^{n-1} \binom{n-1}{j} \\
= \frac{n}{2^n - 1} (n-1) \sum_{l=0}^{n-2} \binom{n-2}{l} + \frac{n}{2^n - 1} 2^{n-1} \\
= \frac{n(n-1)2^{n-2}}{2^n - 1} + \frac{2^{n-1}n}{2^n - 1} \\
= \frac{n^24^{n-1}}{(2^n - 1)^2}.
\]

(1)

If we then computer \( \text{Var}(X) = E[X^2] - E[X]^2 \) we get:

\[
\text{Var}(X) = \frac{n(n-1)2^{n-2}}{2^n - 1} + \frac{2^{n-1}n}{2^n - 1} - \frac{n^24^{n-1}}{(2^n - 1)^2}.
\]

Through some straight forward algebra reductions you get the desired formula above.
Show also that for \( n \) large

\[ \text{Var}(X) \approx \frac{n}{4} \]

in the sense that the ratio of \( \frac{n}{4} \) and \( \text{Var}(X) \) approaches 1 as \( n \to \infty \).

\[
\lim_{n \to \infty} \left( \frac{4}{n} \right) \frac{n \cdot 4^{n-1} - n(n+1)2^{n-2}}{(2^n - 1)^2} = \lim_{n \to \infty} \frac{4^n - (n + 1)2^n}{(2^n - 1)^2} \\
= \lim_{n \to \infty} \frac{2^n - n - 1}{2^n - 2 + 2^{-n}} \\
= \lim_{n \to \infty} \frac{2^n \ln(2) - 1}{2^n \ln(2) - \ln(2)2^{-n}} \\
= \lim_{n \to \infty} \frac{1 - \frac{1}{2^n \ln(2)}}{1 - 4^{-n}} \\
= 1
\]

Where in the third equality I have used L’Hopital’s rule.

**Problem 8.** An urn initially contains one red and one blue ball. At each stage a ball is randomly chosen and then replaced along with another of the same color. Let \( X \) denote the selection number of the first chosen ball that is blue. For instance, if the first selection is red and the second blue, then \( X \) is equal to 2.

(a) Find \( P\{X > i\} \) for \( i \geq 1 \).

(b) Show that with probability 1, a blue ball is eventually chosen.

(c) Find \( E[X] \).
Solution: To solve this problem we first compute the probability $P\{X = i\}$ for $i \geq 1$. To do this we must draw a red on the first $i-1$ draws, and after each draw we add one more red ball. Thus for each draw there is one more total ball, and one more red ball, and when we finally draw the blue ball on the $i$th draw there are $i+1$ total balls to choose from 1 which is blue. Thus,

$$P\{X = i\} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{i-1}{i} \cdot \frac{1}{i+1} = \frac{1}{i(i+1)}.$$  

We could clearly then write:

$$P\{X > i\} = \sum_{k=i+1}^{\infty} \frac{1}{k(k+1)}.$$  

However, we would like to evaluate this sum. Arguing combinatorially the $P\{X > i\}$ is the probability that we draw a blue ball for the first time after the $i$th draw. This is equivalent to the probability that the first $i$ draws are all red. As argued above this must be given by:

$$P\{X > i\} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{i-1}{i} \cdot \frac{i}{i+1} = \frac{1}{i(i+1)}$$

which is the answer to (b). This also implies that $\sum_{k=i+1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{i+1}$. Can we show this algebraically? Yes. To do this define:

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}.$$  

Using the fact that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ we have that:

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

By regrouping we get that:

$$S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$
From this we observe two things. First, we have that: \( P\{X \leq i\} = S_i = \frac{i}{i+1} \). Second by taking the limit of \( S_n \) as \( n \to \infty \) we get 1. This means that the infinite series converges and:

\[
P(\text{we eventually draw a blue ball}) = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.
\]

This answers part \( (b) \). Also since we know have that:

\[
1 = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{i} \frac{1}{k(k+1)} + \sum_{k=i+1}^{\infty} \frac{1}{k(k+1)}
\]

\[
= \frac{i}{i+1} + \sum_{k=i+1}^{\infty} \frac{1}{k(k+1)}
\]

Thus we have shown algebraically that:

\[
\frac{1}{i+1} = 1 - \frac{i}{i+1} = \sum_{k=i+1}^{\infty} \frac{1}{k(k+1)}.
\]

Since we computed that \( P\{X = i\} = \frac{1}{i(i+1)} \), then we have that:

\[
E[X] = \sum_{k=1}^{\infty} k \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k+1}
\]

This series is however divergent, and thus the expected value is infinite.
Problem 9. An urn contains 4 white and 4 black balls. We randomly choose 4 balls. If 2 of them are white and two are black then we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly \( n \) selections? How many selections on average do we make?

Solution: If we let \( X \) be the number of draws until we get the desired situation then \( X \) is a geometric random variable with the parameter \( p \) equal to the probability of drawing exactly 2 white and 2 black balls on your choice of 4. To compute \( p \) you consider all of the possible groups of 2 white balls \( \binom{4}{2} \) and all the possible groups of black balls \( \binom{4}{2} \). Since there are \( \binom{8}{4} \) possible groups of 4 balls to choose from then:

\[
p = \frac{\binom{4}{2} \binom{4}{2}}{\binom{8}{4}} = \frac{18}{35}.
\]

Consequently the probability that we make exactly \( n \) selections is given by:

\[
P\{X = n\} = \left(\frac{17}{35}\right)^{n-1} \frac{18}{35}.
\]

The expected value for a geometric random variable is given by \( \frac{1}{p} \) and thus on average we can expect to make \( E[X] = \frac{35}{18} \) draws.