Problem 1. Show that
\[ E[Y] = \int_0^\infty P\{Y > y\}dy - \int_0^\infty P\{Y < -y\}dy. \]

Problem 2. Let \( X \) be a continuous random variable. Using the definition of expectation value for continuous random variables verify that:

(a) \( E[aX + b] = aE[X] + b \) for constants \( a, b \)

(b) \( \text{Var}(X) = E[X^2] - (E[X])^2 \)

Problem 3. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M.

(a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A?

(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 A.M.?

Problem 4. A point is chosen at random on a line segment of length \( L \). Interpret this statement and find the probability that the ratio of the shorter to the longer segment is less than \( \frac{1}{4} \).

Problem 5. The speed of a molecule in a uniform gas at equilibrium is a random variable whose probability density function is given by
\[
 f(x) = \begin{cases} 
 a x^2 e^{-bx^2} & x \geq 0 \\
 0 & x < 0 
\end{cases}
\]
where \( b = \frac{m}{2kT} \) and \( k, T, \) and \( m \) denote, respectively, Boltzmann’s constant, the absolute temperature, and the mass of the molecule. Evaluate \( a \) in terms of \( b \). And compute the expected value of \( X \) in terms of \( b \).
Problem 6. Let $X$ be a random variable that takes on values between 0 and $c$. That is $P\{0 \leq X \leq c\} = 1$. Show that

$$Var(X) \leq \frac{c^2}{4}.$$ 

Problem 7. From a set of $n$ elements a nonempty subset is chosen at random in the sense that all of the nonempty subsets are equally likely to be selected. Let $X$ denote the number of elements in the chosen subset. Show that:

$$E[X] = \frac{n}{2 - \left(\frac{1}{2}\right)^{n-1}}$$

$$Var(X) = \frac{n \cdot 4^{n-1} - n(n+1)2^{n-2}}{(2^n - 1)^2}$$

Show also that for $n$ large

$$Var(X) \approx \frac{n}{4}$$

in the sense that the ratio of $\frac{n}{4}$ and $Var(X)$ approaches 1 as $n \to \infty$.

Problem 8. An urn initially contains one red and one blue ball. At each stage a ball is randomly choosen and then replaced along with another of the same color. Let $X$ denote the selection number of the first chosen ball that is blue. For instance, if the first selection is red and the second blue, then $X$ is equal to 2.

(a) Find $P\{X > i\}$ for $i \geq 1$.

(b) Show that with probability 1, a blue ball is eventually chosen.

(c) Find $E[X]$.

Problem 9. An urn contains 4 white and 4 black balls. We randomly choose 4 balls. If 2 of them are white and two are black then we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly $n$ selections? How many selections on average do we make?