Homework 4 (due October 20, 2009)

Problem 1. There is a 50% chance that the queen carries the gene for hemophilia. If she is a carrier, then each prince has a 50% chance of having hemophilia.

(a) If the queen has had three princes without the disease, what is the probability that the queen is a carrier?

Let $Q$ be the event that the Queen is a carrier. Let $E_i$ be the event that that prince $i$ has the disease. We seek $P(Q|E_1^c E_2^c E_3^c)$. Using Bayes’ formula we have:

$$P(Q|E_1^c E_2^c E_3^c) = \frac{P(E_1^c E_2^c E_3^c|Q)P(Q)}{P(E_1^c E_2^c E_3^c|Q)P(Q) + P(E_1^c E_2^c E_3^c|Q^c)P(Q^c)}$$

Using the information we are given in the problem we deduce that $P(Q) = P(Q^c) = .5$, and that $P(E_1^c E_2^c E_3^c|Q) = (.5)^3 = \frac{1}{8}$, and $P(E_1^c E_2^c E_3^c|Q^c) = 1$. Putting this info into the equation above we have:

$$P(Q|E_1^c E_2^c E_3^c) = \frac{.5(\frac{1}{8})}{.5(\frac{1}{8}) + 1} = \frac{1}{9}.$$

(b) If there is a fourth prince, what is the probability that he will have hemophilia?

Here we want $P(E_4|E_1^c E_2^c E_3^c) = P(Q|E_1^c E_2^c E_3^c)P(E_4|Q E_1^c E_2^c E_3^c)$. We computed $P(Q|E_1^c E_2^c E_3^c) = \frac{1}{9}$, and $P(E_4|Q E_1^c E_2^c E_3^c) = \frac{1}{2}$ since the fourth prince having it only depends on if the Queen has it, not on if the other princes have it. Thus $P(E_4|E_1^c E_2^c E_3^c) = \frac{1}{18}$.

Problem 2. A and B are involved in a duel. The rules of the duel are that A and B are to pick up their guns and shoot at each other simultaneously. If one or both are hit, then the duel is over. If both shots miss, then they repeat the process. Suppose that the results of the shots are independent and that each shot of A will hit B with probability $p_A$, and each shot of B will hit A with probability $p_B$. What is

(a) the probability that A is not hit;
If I consider the probability that the dual ends on the kth trial then the first \( n-k \) trials both A and B are not hit and then on the last trial A is not hit and B is. This happens with probability:

\[(1 - p_A)^{k-1}(1 - p_B)^{k-1}p_A(1 - p_B).\]

Since the dual can end on any number of duals from 1 to infinity then we must add up all the cases to get:

\[
\sum_{k=1}^{\infty} (1 - p_A)^{k-1}(1 - p_B)^{k-1}p_A(1 - p_B) = \frac{p_A(1 - p_B)}{1 - (1 - p_A)(1 - p_B)}.
\]

(b) the probability that both duelists are hit;
Similar arguments give that this probability is:

\[
\frac{p_A p_B}{1 - (1 - p_A)(1 - p_B)}.
\]

(c) the probability that the duel ends exactly after the \( n \)th round of shots;
The probability that it ends on the \( n \)th round of shots is equivalent to no one getting hit on the first \( n-1 \) plays and then that someone gets hit on the \( n \)th hit which is equal to \((1 - (1 - p_A)(1 - p_B))\). Thus the probability is \((1 - (1 - p_A)(1 - p_B))(1 - p_A)^{n-1}(1 - p_B)^{n-1}\).

(d) the conditional probability that the duel ends after the \( n \)th round of shots given that A is not hit;
One defines the event L as the event that A is not hit, and N as the event that the dual ends on the \( n \)th round.

\[
P(N|L) = \frac{P(LN)}{P(L)} = \frac{p_A(1 - p_B)(1 - p_A)^{n-1}(1 - p_B)^{n-1}}{1 - (1 - p_A)(1 - p_B)} = \frac{(1 - p_A)^{n-1}(1 - p_B)^{n-1}}{1 - (1 - p_A)(1 - p_B)}.
\]

(e) the conditional probability that the duel ends after the \( n \)th round of shots given that both duelists are hit?
Similar arguments to (d) give the same answer.

**Problem 3.** An engineering system consisting of \( n \) components is said to be a \( k \)-out-of-\( n \) system \((k \leq n)\) if the system functions if and only if at least \( k \) of the \( n \) components function. Suppose that all components function independently of each other.
(a) If the i-th component functions with probability \( P_i \) for \( i = 1, 2, 3, 4 \), compute the probability that a 2 out of 4 system functions.

For this you must consider the three separate cases where exactly 2, 3, or 4 components work. Or consider the cases where exactly 0 or 1 work and subtract their total from 1. This is what I choose to do here. Exactly 0 function with probability \((1 - P_1)(1 - P_2)(1 - P_3)(1 - P_4)\), and exactly 1 functions with probability \((P_1)(1 - P_2)(1 - P_3)(1 - P_4) + (1 - P_1)(1 - P_2)(1 - P_3)P_4 + (1 - P_1)(1 - P_2)P_3(1 - P_4) + (1 - P_1)P_2(1 - P_3)(1 - P_4)\).

Consequently the probability that the system works is:

\[
1 - (1 - P_1)(1 - P_2)(1 - P_3)(1 - P_4) - \left\{ \begin{array}{c}
(P_1)(1 - P_2)(1 - P_3)(1 - P_4) + (1 - P_1)P_2(1 - P_3)(1 - P_4) \\
+ (1 - P_1)(1 - P_2)(1 - P_3)P_4 + (1 - P_1)(1 - P_2)P_4(1 - P_3) \\
+ (1 - P_1)P_2P_3(1 - P_4) + (1 - P_1)P_2P_4(1 - P_3) + (1 - P_1)(1 - P_2)P_3P_4 \\
+ (1 - P_1)(1 - P_2)P_3P_4 + (1 - P_1)(1 - P_2)(1 - P_3)P_4 \\
+ (1 - P_1)(1 - P_2)(1 - P_3)P_4 + (1 - P_1)(1 - P_2)(1 - P_3)(1 - P_4)
\end{array} \right\}.
\]

(b) Repeat part (a) for a 3-out-of-5 system.

In a similar fashion this system has a probability of functioning with probability:

\[
1 - (1 - P_1)(1 - P_2)(1 - P_3)(1 - P_4)(1 - P_5) - \left\{ \begin{array}{c}
(P_1)(1 - P_2)(1 - P_3)(1 - P_4)(1 - P_5) + (1 - P_1)P_2(1 - P_3)(1 - P_4)(1 - P_5) \\
+ (1 - P_1)(1 - P_2)(1 - P_3)(1 - P_4)P_5 + (1 - P_1)(1 - P_2)P_3(1 - P_4)(1 - P_5) + (1 - P_1)(1 - P_2)P_4(1 - P_3)(1 - P_5) \\
+ (1 - P_1)(1 - P_2)(1 - P_3)(1 - P_4)(1 - P_5) + (1 - P_1)(1 - P_2)(1 - P_3)P_4(1 - P_5) + (1 - P_1)(1 - P_2)P_3P_4(1 - P_5) + (1 - P_1)(1 - P_2)(1 - P_3)P_4P_5 \\
+ (1 - P_1)(1 - P_2)(1 - P_3)P_4P_5 + (1 - P_1)(1 - P_2)(1 - P_3)(1 - P_4)P_5 \\
+ (1 - P_1)(1 - P_2)(1 - P_3)(1 - P_4)(1 - P_5)
\end{array} \right\}.
\]

(c) Repeat for a k-out-of-n system when all the \( P_i \) equal \( p \).

This is similar, but now since the probabilities are all the same we just must count all of the events that have exactly \( i \) working, which is \( \binom{n}{i} \), with each of these occurring with probability \( p^i(1 - p)^{n-i} \). Thus the probability of exactly \( i \) working is \( \binom{n}{i}p^i(1 - p)^{n-i} \). Since we are interested in the probability that \( k \) or more work, then we sum the probability for exactly \( i \) working from \( i = k \) to \( i = n \):

\[
\sum_{i=k}^{n} \binom{n}{i}p^i(1 - p)^{n-i}.
\]

**Problem 4.** In successive rolls of a pair of dice, what is the probability that there will appear 2 sevens before 6 even numbers. (Here we are talking about the sum on the dice).
This is a tricky problem that I will be happy to explain to anyone, but basically it comes down to the problem of points, but one thing is the probabilities to be if I know I have rolled either a 7 or and even, what is the probability that it is a 7. This is equal to \( \frac{1}{4} \). Thus the probability that I get 2 sevens before I get 6 evens is equal to:

\[
\sum_{k=2}^{7} \binom{7}{k} \left( \frac{1}{4} \right)^k \left( \frac{3}{4} \right)^{7-k}
\]

**Problem 5.** A and B play a series of games. Each game is independently won by A with probability p and by B with probability 1-p. They stop when the total number of wins of one of the players is greater than that of the other player. The player with the greater number of total wins is declared the winner.

(a) Find the probability the 4 games are played.

If 4 games are to be played there are 4 ways that this can happen either ABAA, ABBB, BAAA, BABB. If we compute the probability for each of these then we get: 
\[ 2p^3(1-p) + 2(1-p)^3p. \]

(b) Find the probability that A is the match winner.

First observe that there can not be a match that ends over an odd number of rounds. If A is the winner in 2k rounds then at the 2k-2nd round A and B must be tied. To be tied then we must have an outcome that looks like ABABAB...AB or ABBABAABBA... or BABABA...BA, but each of these events looks like \((AB)(AB)...(AB)\) where the A and B can be ordered in any order among the k-1 pairs. So there are \(2^{k-1}\) different orderings of this sort and the probability of each of these events is \((p(1-p))^{k-1}\). Then for A to win the last two rounds must be won by A which is \(p^2\), thus the probability that A wins on the 2kth round is given by \(2^{k-1}(p(1-p))^{k-1}p^2 = 2^{k-1}p^{k+1}(1-p)^{k-1}\) To get the probability that A wins overall we must add up these cases from k=1 to \(\infty\) or:

\[
\sum_{k=1}^{\infty} 2^{k-1}p^{k+1}(1-p)^{k-1} = p^2 \sum_{k=1}^{\infty} (2p(1-p))^{k-1}
\]

Now it is a calculus exercise to check that the maximum of \(p(1-p)\) is 1/4 so we have that \(2p(1-p) \leq 1/2\), thus we again have a geometric series that can be summed giving:

\[
p^2 \sum_{k=1}^{\infty} (2p(1-p))^{k-1} = \frac{p^2}{1 - 2p(1-p)} = \frac{p^2}{1 - 2p + 2p^2}
\]

**Problem 6.** Let \(X\) and \(Y\) be random variables taking values in the set of natural numbers, \(\mathbb{N} = \{1, 2, 3, \ldots\}\), whose probability mass functions are of the form given in (a) and (b), respectively. Find the values of the constants \(C_1\) and \(C_2\).
(a) \( p_X(k) = C_1 2^{-k}, \ k = 1, 2, \ldots \)

(b) \( p_Y(k) = C_2 2^{-k} / k, \ k = 1, 2, \ldots \)

**Hint:** (b) Use the Taylor series of \( \ln(1 + x) \).

**Solution** To solve for \( C_1 \) one uses the fact that:

\[
1 = \sum_{k=1}^{\infty} p_X(k) = C_1 \sum_{k=1}^{\infty} 2^{-k} = C_1 (1)
\]

Where the last equality is found using the fact that the series is a summable geometric series that is equal to 1. Thus \( C_1 = 1 \). For \( C_2 \) we have by the same logic that:

\[
1 = \sum_{k=1}^{\infty} p_Y(k) = C_2 \sum_{k=1}^{\infty} \frac{2^{-k}}{k} = C_2 (\ln(2))
\]

The last equality came from recognizing that:

\[
\ln(1 + x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}
\]

Evaluating this at \( x = -\frac{1}{2} \) we have that, \(-\ln(\frac{1}{2}) = \sum_{k=1}^{\infty} \frac{2^{-k}}{k}. \) Thus, we can conclude that \( C_2 = \frac{1}{\ln(2)} \).

**Problem 7.** In the previous problem, find

(a) \( P(X > 1) \);

This is equal to \( 1 - P(X = 1) = 1 - 2^{-1} = \frac{1}{2} \).

(b) \( P(X \text{ is even}) \);

\[
= \sum_{k=1}^{\infty} p_X(2k) = \sum_{k=1}^{\infty} 2^{-2k} = \sum_{k=1}^{\infty} 4^{-k} = \frac{1}{3}
\]

Where the last term is found by summing the geometric series.

(c) \( P(Y > 1) \);

\[
= 1 - P(Y = 1) = 1 - \frac{2^{-1}}{\ln(2)} = 1 - \frac{1}{\ln(4)}.
\]

(d) \( P(Y \text{ is even}) \).

\[
= \sum_{k=1}^{\infty} p_Y(2k) = \frac{1}{\ln(2)} \sum_{k=1}^{\infty} \frac{2^{2k}}{2k} = \frac{1}{\ln(4)} \sum_{k=1}^{\infty} \frac{4^{-k}}{k} = \frac{\ln(\frac{1}{3})}{\ln(4)}.
\]