Math 351: Midterm 2

Name: ____________________________________________

Instructions:

- Explain all of your answers! Answers without explanation will not receive credit.
- State explicitly all assumptions made! Make sure they are reasonable.
- Use the back of the page if you run out of room—but label clearly where you’ve solved each problem.
- Give a numerical answer to four decimal places.

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1. (15 points) Identify the requested information about the random variable you would use in each of the following questions. Indicate the parameters where possible. You do not need to answer the question!

Whenever a continuous random variable could reasonably be used, use it.

- Example: The waitstaff are extremely distracted and have a 3 chance of making a mistake with your coffee order, giving you decaf even though you order caffeinated. If you order four caffeinated coffees, what are the odds that you get exactly two decaf and two regular coffees?
  
  (i) \( X = \) the number of decaf coffees you receive.
  
  (ii) \( X \) is a \textit{binomial} random variable.
  
  (iii) The parameters of \( X \) are \((4,.3)\).

(a) (3 points) Scientists have calculated that on average there is an earthquake that registers 5 or higher on the Richter scale once every 6 years. How long could I expect to wait until the next earthquake that registers 5 or higher?

  i. \( X = \) the length of time in years until the next earthquake that registers 5 or higher.
  
  ii. \( X \) is a \underline{ } random variable.
  
  iii. The parameters of \( X \) are \underline{ }.

(b) (3 points) On average once every 6 years classes at SU are cancelled due to weather, we call these days “snow days”. How many snow days do you expect over the course of your four years at the University?

  i. \( X = \) the number of snow days over your stay at the University.
  
  ii. \( X \) is a \underline{ } random variable.
  
  iii. The parameters of \( X \) are \underline{ }.

(c) (3 points) Given the same setup as the example, how many coffees would you need to order to receive two regular coffees?

  i. \( X = \) the number of coffees necessary to receive two regular coffees.
  
  ii. \( X \) is a \underline{ } random variable.
  
  iii. The parameters of \( X \) are \underline{ }.

\( continued \ on \ next \ page \)
(d) (3 points) I have 80 students in my probability class this semester. I am in a particularly giving sort of mood, so I decide to give out some bonus points. I decide I am going to reward 100 bouns points total, and I award them in the following way. I put all the students names in a hat and then draw one. That student gets an extra point. I put the students name back in the hat and I repeat 100 times. What is the problem that one particular student (insert your favorite classmates name) gets at least 3 bouns points?

i. $X =$ the number of extra points a particular student receives.

ii. $X$ is a ________________________________ random variable.

iii. The parameters of $X$ are ________________________________.

(e) (3 points) A train heading to Seattle from Portland departs sometime every half hour, but due to delays and other trains, its equally likely to depart anytime in the half an hour. If you want to catch a train to Portland and you arrive at 12, what is the probability that you have to wait more then 15 minutes for the train?

i. $X =$ the length of time in hours that you wait for the train to depart.

ii. $X$ is a ________________________________ random variable.

iii. The parameters of $X$ are ________________________________.
2. (12 points) Suppose that the life span in days of a certain type of battery is described by the random variable $X$ which has density:

$$f(x) = \begin{cases} 
2xe^{-x^2+1} & 1 \leq x < \infty \\
0 & \text{otherwise}
\end{cases}$$

(a) (6 points) What is the probability that one battery will still function after 1.5 days?

Here we want to find $P(X > 1.5)$.

$$P(X > 1.5) = \int_{1.5}^{\infty} 2xe^{-x^2+1} \, dx$$

Using a $u$ substitution with $u = -x^2 + 1$, and $du = -2xdx$ then we have that:

$$P(X > 1.5) = \int_{x=1.5}^{\infty} -e^u \, du = -e^u \bigg|_{x=1.5}^{\infty} = -e^{-x^2+1} \bigg|_{1.5}^{\infty} = 0 - (-e^{-1.5^2+1}) = e^{1-1.5^2} \approx 0.286505$$

(b) (6 points) Suppose that you use the batteries constantly. When one fails you immediately replace it with another one. Your boss rewards you with a bonus of $100 each time a new battery lasts longer then 1.5 days. What is the probability that you will go through at least 6 batteries before you get $200?

There are a couple ways of doing this problem one of them is to think of the variable $Y = \text{the number of batteries required until the second one lasting more than 1.5 days}$. This is a negative binomial random variable with parameters $p = .286505$ and $r = 2$. Thus it’s probability mass function is given by:

$$p(k) = \binom{k-1}{1}(.286505)^2(1-.286505)^{k-2}$$

We want to compute the probability that $Y \geq 6$. So,

$$P(Y \geq 6) = 1 - P(Y \leq 5) = 1 - (p(2) + p(3) + p(4) + p(5))$$

$$= 1 - ((1(0.286505)^2)(0.713495) + 2(0.286505)^2(0.713495)^1 + 3(0.286505)^2(0.713495)^2 + 4(0.286505)^2(0.713495)^3)$$

$$\approx 0.556158$$
3. (8 points) Calls coming into a certain call center behave according to a Poisson process and an average of 20 calls come in per hour. On Mondays, Mary is the only operator answering calls and she has to step away from the phone for 30 minutes while she gets lunch. What is the probability that she misses at least 8 calls while she is gone?

Let \( X \) = The number of calls during a 30 minute period, since this is a Poisson process then \( X \) is Poisson with a parameter \( \lambda = 10 \) (since we are dealing with half an hour). The probability mass functions is then given by 
\[
P(x) = \frac{e^{-10}(10)^x}{x!}
\]
We then seek the probability \( P(X \geq 8) \).

Thus,
\[
P(X \geq 8) = 1 - P(X \leq 7)
\]
\[
= 1 - \left( \frac{e^{-10}(10)^0}{0!} + \frac{e^{-10}(10)^1}{1!} + \frac{e^{-10}(10)^2}{2!} + \frac{e^{-10}(10)^3}{3!} + \frac{e^{-10}(10)^4}{4!} + \frac{e^{-10}(10)^5}{5!} + \frac{e^{-10}(10)^6}{6!} + \frac{e^{-10}(10)^7}{7!} \right)
\]
\[
\approx .7798
\]

4. (8 points) Let \( X \) be a normal random variable with parameters \( \mu = 2 \) and \( \sigma = 1 \). If \( Y = X^2 \) what is the probability density function of \( Y \)? Be sure to include the range of values for which \( Y \) is nonzero!

We start by writing the cdf for \( Y \).

For \( y > 0 \) we have:
\[
F_Y(y) = P(Y \leq y) = P(X^2 \leq y)
\]
\[
= P(-\sqrt{y} \leq X \leq \sqrt{y})
\]
\[
= F_X(\sqrt{y}) - F_X(-\sqrt{y})
\]

Now we can differentiate with respect to \( y \) giving:
\[
\frac{d}{dy} [F_Y(y)] = \frac{d}{dy} [F_X(\sqrt{y})] = \frac{d}{dy} [F_X(-\sqrt{y})]
\]
\[
f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})
\]

Since \( X \) is normal with \( \mu = 2 \) and \( \sigma = 1 \) then we have that \( f_X(x) = (1/\sqrt{2\pi}) e^{(x-2)^2/2} \). Putting this into the formula we then have that for \( y \geq 0 \),
\[
f_Y(y) = \frac{1}{2\sqrt{2\pi y}} e^{-(\sqrt{y}-2)^2} + \frac{1}{2\sqrt{2\pi y}} e^{-(\sqrt{y}+2)^2} = .
\]
\[
f_Y(y) = 0 \text{ if } y < 0.
\]
5. (12 points) Suppose that \( X \) and \( Y \) are jointly continuous and have a joint density function given by:

\[
f(x, y) = \begin{cases} 
cxye^{-y} & 0 < x < 2, 0 < y < 2 - x \\
0 & \text{otherwise}
\end{cases}
\]

(a) (6 points) What is the marginal density for \( X \)? (Compute all integrals but leave your answer in terms of \( c \)).

The marginal density for \( X \) is given by:

\[
f_X(x) = \int_0^{2-x} cxye^{-y} \, dy = cx \int_0^{2-x} ye^{-y} \, dy
\]

To compute the above integral we use integration by parts letting \( u = y \) and \( dv = e^{-y} \, dy \). This gives:

\[
f_X(x) = cx \left[ -ye^{-y} \bigg|_0^{2-x} + \int_0^{2-x} e^{-y} \, dy \right] = cx \left[ -ye^{-y} \bigg|_0^{2-x} - e^{-y} \bigg|_0^{2-x} \right]
\]

Thu for \( 0 < x < 2 \),

\[
f_X(x) = cx \left( (x - 2)e^{x-2} - e^{-x-2} + 1 \right) = cx(x - 3)e^{x-2} + cx
\]

(b) (3 points) Set up but do not evaluate an integral equal to \( P(Y > 1/2 - X) \)?

\[
P(Y > 1/2 - X) = \int_0^{1/2} \int_{1/2-x}^{2-x} cxye^{-y} \, dy \, dx + \int_{1/2}^{2} \int_0^{2-x} cxye^{-y} \, dy \, dx
\]

or

\[
P(Y > 1/2 - X) = 1 - \int_0^{1/2} \int_0^{1/2-x} cxye^{-y} \, dy \, dx
\]

(c) (3 points) Set up but do not evaluate a ratio of integrals equal to \( P(X < 1 \mid Y < 1) \).

\[
P(X < 1 \mid Y < 1) = \frac{P(X < 1, Y < 1)}{P(Y < 1)} = \frac{\int_0^1 \int_0^1 cxye^{-y} \, dy \, dx}{\int_0^1 \int_0^{2-y} cxye^{-y} \, dy \, dx}
\]
6. (12 points) Patrick McNeal is a noted professional dart thrower. Dart throwing competitions score roughly as follows: the thrower receives 20 points if his/her shot is within 1 inch of the center, 5 points if it is between 1 and 3 inches from the center, and 3 points if it is between 3 and 5 inches from the center and 0 if it is more then 5 inches away. Suppose that the distance from Patrick’s shot to the target is uniformly distributed between 0 and 10 inches from the center. Let $X$ be the number of points Patrick receives on one throw.

(a) (6 points) What is the mean and variance of $X$?

First we calculate the probability that we get each score. In particular $P(X = 20) = \int_0^1 \frac{1}{10} \, dx = \frac{1}{10}$. Similarly we can calculate that the pmf for $X$ is:

$$p(X = i) = \begin{cases} \frac{1}{2}, & i = 0 \\ \frac{1}{5}, & i = 3 \\ \frac{1}{5}, & i = 5 \\ \frac{1}{10}, & i = 20 \end{cases}$$

From this we obtain that:

$$E[X] = 0p(0) + 3p(3) + 5p(5) + 20p(20) = \frac{3}{5} + \frac{5}{5} + \frac{20}{10} = 3.6$$

$$E[X^2] = 0^2p(0) + 3^2p(3) + 5^2p(5) + 20^2p(20) = \frac{9}{5} + \frac{25}{5} + \frac{400}{10} = 46.8$$

So the mean is 3.6 and the variance is $E[X^2] - E[X]^2 = 46.5 - (3.6)^2 = 33.84$.

(b) (6 points) What is the cumulative density function for $X$? (give a piecewise definition and sketch a rough graph).

The cdf is given by:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & 0 \leq x < 3 \\ \frac{7}{10}, & 3 \leq x < 5 \\ \frac{9}{10}, & 5 \leq x < 20 \\ 1, & x \geq 20 \end{cases}$$
7. (8 points) Twelve percent of the population is left-handed. Approximate the probability that between 480 and 510 of the 4000 students at Seattle University are left-handed. State clearly all assumptions made in your model.

For this problem, I assume that 480 and 510 are not to be included. In other words we want to be strictly between 480 and 510 students. If we let $X$ be the number of left handed students at SU then assuming that each student is left handed independently of any other student with probability .12, then this variable would be binomial with $n = 4000$ and $p = .12$, However since $4000(.12)(.88) = 422.4 > 10$ we can use the Demoivre-Laplace Theorem and approximate the binomial random variable with a normal random variable. We start though by including a continuity correction.

$$P(480 < X < 510) = P(480.5 < X < 509.5)$$
$$= P\left( \frac{480.5 - np}{\sqrt{np(1-p)}} < \frac{X - np}{\sqrt{np(1-p)}} < \frac{509.5 - np}{\sqrt{np(1-p)}} \right)$$
$$= P\left( \frac{480.5 - 480}{\sqrt{480(.88)}} < \frac{X - 480}{\sqrt{480(.88)}} < \frac{509.5 - 480}{\sqrt{480(.88)}} \right)$$
$$\approx \Phi\left( \frac{29.5}{20.5524} \right) - \Phi\left( \frac{.5}{20.5524} \right)$$
$$= \Phi(1.43536) - \Phi(0.0243281)$$
$$\approx .9251 - .5080 = .4171$$