For each of the following, fill in the missing information to make the following series equivalent. Then, write the first few terms in the series for each side to make sure that you have found the correct expression.

For example, if you're given

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a\_\_\_ x\text{———},$$

you should fill in the empty spots as follows

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^{n-1}.$$

You can then justify that the relationships are equivalent by writing out the first few terms in each series. Thus,

$$a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots = a_{1-1} x^{1-1} + a_{2-1} x^{2-1} + a_{3-1} x^{3-1} + \ldots \quad \checkmark.$$

1. $\displaystyle\sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} = \sum_{n=0}^{\infty} a_{n+1} \cdot (n+1) \cdot x^{n}$

   up by 1!   *down by 1*

   Note: if the start of the sum goes ↓ 1 the indices inside go ↑ 1!

2. $\displaystyle\sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n-2} = \sum_{n=0}^{\infty} a_{n+2} \cdot (n+2) \cdot (n+1) \cdot x^{n}$

   up by 2   *down by 2*

3. $\displaystyle\sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n+1} = \sum_{n=2}^{\infty} a_{n-1} \cdot (n-1) \cdot x^{n}$

   down by 1   *up by 1*

Don't turn over the page just yet...

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4. Consider the initial value problem

$$P(x)y'' + Q(x)y' + R(x)y = 0, \qquad y(0) = y_0, y'(0) = y_0'.$$

Assume that $P(x)$, $Q(x)$, and $R(x)$ are all polynomials and have no common factors.

(a) Write the above differential equation in the standard form for second-order linear differential equations

$$y'' + p(x)y' + q(x)y = 0.$$

What are the function $p(x)$ and $q(x)$ in terms of $P(x)$, $Q(x)$, and $R(x)$?

$$\boxed{p(x) = \frac{Q(x)}{P(x)}} \qquad \boxed{q(x) = \frac{R(x)}{P(x)}}$$

(b) Using the fact that $y''(x) = -p(x)y'(x) - q(x)y(x)$, for all x, plug in $x = 0$. How is $y''(0)$ related to y_0 and y_0'?

$$y''(0) = -p(0)y'(0) - q(0)y(0)$$
$$\boxed{y''(0) = -\left(p(0)y_0' + q(0)y_0\right)} \quad \text{(A)}$$

also note: for a series,
$$y_0 = a_0 \text{ and } y_0' = a_1.$$

(c) Consider the series solution for the above problem centered at $x = 0$. Using the above relationship in conjunction with the series representation for $y(x)$ evaluated at $x = 0$, find a_2 in terms of $p(0)$, $q(0)$, a_0 and a_1. If $y(x) = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$

Note: $y''(x) = a_2 \cdot 2 \cdot 1 + a_3 \cdot 3 \cdot 2 \cdot x + a_4 \cdot 4 \cdot 3 \cdot x^2 + \dots \rightarrow \boxed{y''(0) = 2a_2}$ (B)

Comparing (A) and (B)

$$2a_2 = -(p(0)a_1 + q(0)a_0) \rightarrow \boxed{a_2 = -\frac{1}{2}\left(p(0)a_1 + q(0)a_1\right)}$$

(d) Show that if $y(x)$ solves the ODE in std. form, then

$$y'''(x) = \frac{d}{dx}\left(-p(x)y'(x) - q(x)y(x)\right).$$

Repeat the above analysis to find $y'''(0)$ in terms of p, q, and a_j for $j = 0, 1, 2$. Use this to solve for a_3.

$$y'''(x) = -\left(p'(x)y'(x) + p(x)y''(x) + q'(x)y(x) + q(x)y'(x)\right)$$
$$y'''(0) = 3 \cdot 2 \cdot a_3 = -\left(p'(0)a_1 + p(0)2a_2 + q'(0)a_0 + q(0)a_1\right)$$
$$\boxed{a_3 = -\frac{1}{6} \cdot \left(p'(0)a_1 + 2p(0)a_2 + q'(0)a_0 + q(0)a_1\right)}$$

(e) Under what conditions on $p(x)$ and $q(x)$ do you think that the series for $y(x)$ given in (a) will converge?

$q(x)$ & $p(x)$ must have all of its derivatives defined at $x = 0$.