MATH 2340 WARM-UP PROBLEMS

1. Consider the second-order differential equation

$$y'' + p(x)y' + q(x)y = g(x). \qquad (1)$$

Under what special circumstance is (1) a *homogeneous* ODE? What about *linear*?

If $g(x) = 0$, then (1) is homogeneous. No matter the values of $p(x)$, $q(x)$, or $g(x)$, the ODE is linear.

2. Now, consider the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(0) = b_1, \;\; y'(0) = b_2.$$

Assume that the general solution to the above differential equation is given by $y(x) = c_1 y_1(x) + c_2 y_2(x)$ where both $y_1(x)$ and $y_2(x)$ both solve the ODE and c_1 and c_2 are arbitrary constants.

Write a system of two equations involving the initial conditions that allows you to solve for the constants c_1 and c_2. If possible, write your equations in matrix form $\mathbf{A}\,\vec{\mathbf{c}} = \vec{\mathbf{b}}$ where A is a 2×2 matrix, and $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are 2×1 vectors.

$$\left.\begin{aligned} y(0) = b_1 &\rightarrow c_1 y_1(0) + c_2 y_2(0) = b_1 \\ y'(0) = b_1 &\rightarrow c_1 y_1'(0) + c_2 y_2'(0) = b_2 \end{aligned}\right\} \longrightarrow \begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

↳ note: if we considered more general initial conditions $y(x_0) = b_1$, $y'(x_0) = b_2$ then we would have the system $\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

3. What must be true about the expression derived above in order to solve for c_1 and c_2 given any set of initial conditions (any value for b_1 and b_2)?

The matrix $\begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix}$ must be invertible (have a unique inverse).

In the more general case mentioned above, we would need the matrix $\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$ to be invertible. Another way to state this is to ensure that the determinant $y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0$.

That's it. No more fun on the other side.