MATH 2340 WARM-UP PROBLEMS

1. What are the general solutions to the following differential equations:

   (a) $y' = 7y$ $\quad \frac{dy}{dx} = 7y \rightarrow y = ce^{7x}$

   (b) $y' + 3y = 0 \rightarrow y = ce^{-3x}$

   (c) $8y' + 4y = 0 \rightarrow y = ce^{-\frac{1}{2}x}$

   notice the pattern! Each ODE is linear, constant coefficient and homogeneous; their correspond solutions all take the form $y = ce^{\lambda x}$ where λ = some constant!

2. Consider the differential equation

$$y'' - 7y' + 12y = 0.$$

   (a) Classify the differential equation.

   2nd order, linear, constant coefficient. homogeneous ($g(x)=0$) ODE.

   (b) Verify that $y_1(x) = e^{3x}$ and $y_2(x) = e^{4x}$ are both solutions to the differential equation.

   Check: $y_1 = e^{3x}$

   $y_1' = 3e^{3x}$

   $y_1'' = 9e^{3x}$

   $9e^{3x} - 7\cdot(3e^{3x}) + 12(e^{3x}) = 0$ ✓

   $y'' - 7y' + 12y = 0$

   $y_2 = e^{4x}$

   $y_2' = 4e^{4x}$

   $y_2'' = 16e^{4x}$

   $16e^{4x} - 7(4e^{4x}) + 12e^{4x} = 0$ ✓

   $y'' - 7y' + 12y = 0$

   (c) Under what conditions is $y_3(x) = c_1e^{3x} + c_2e^{4x}$ also a solution to the differential equation?

   Plugging in $y_3 = c_1e^{3x} + c_2e^{4x}$ we find:

$$9c_1e^{3x} + 16c_2e^{4x} - 7(3c_1e^{3x} + 4c_2e^{4x}) + 12(c_1e^{3x} + c_2e^{4x}) = 0 \checkmark$$

   As long as c_1 and c_2 are constants, $y = c_1e^{3x} + c_2e^{4x}$ is also a solution.

   Take away: linear combos of solutions to linear, 2nd order, constant coeff., homog, ODES are also solutions.

Please turn over the page for more fun! ⟶

3. Consider the differential equation

$$r(x)y'' + p(x)y' + q(x)y = 0.$$

(a) Classify the differential equation.

linear, 2^{nd} order, homogeneous ODE.

(b) Assume that $y_1(x)$ and $y_2(x)$ are both solutions to the above differential equation. Under what conditions is $y_3(x) = c_1y_1(x) + c_2y_2(x)$ also a solution?

c_1 and c_2 are constant!

$\boxed{\text{Check}}$: $y_3 = c_1y_1 + c_2y_2$, $y_3' = c_1y_1' + c_2y_2'$ $y_3'' = c_1y_1'' + c_2y_2''$

$$r(x)\left[c_1y_1'' + c_2y_2''\right] + p(x)\left[c_1y_1' + c_2y_2'\right] + q(x)\left[c_1y_1 + c_2y_2\right] \overset{?}{=} 0$$

Group the c_1's and c_2's.

$$c_1\underbrace{\left[r(x)y_1'' + p(x)y_1' + q(x)y_1\right]}_{\text{this is zero b/c } y_1 \text{ is a solution}} + c_2\underbrace{\left[r(x)y_2'' + p(x)y_2' + q(x)y_2\right]}_{\text{this is zero b/c } y_2 \text{ is a solution.}} \overset{?}{=} 0$$

$$c_1 \cdot 0 + c_2 \cdot 0 = 0 \checkmark$$

Yes! $y_3 = c_1y_1 + c_2y_2$ solves $r(x)y'' + p(x)y' + q(x)y = 0$ provided y_1 and y_2 both solve $r(x)y'' + p(x)y' + q(x)y = 0$.

(c) What do you think is special about the differential equation $r(x)y'' + p(x)y' + q(x)y = 0$ that allows the above statement to be true.

Linear and Homogeneous