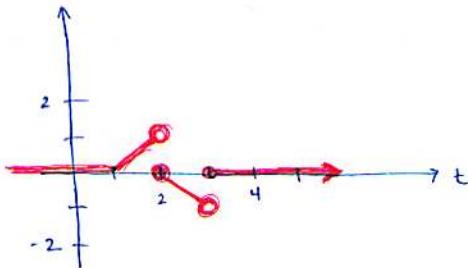


MATH 2340 WARM-UP PROBLEMS

1. Consider the function

$$f(t) = \begin{cases} 0 & \text{if } t < 1 \\ t - 1 & \text{if } 1 \leq t < 2 \\ -t + 2 & \text{if } 2 \leq t < 3 \\ 0 & \text{if } t \geq 3. \end{cases}$$

- (a) Sketch a graph of $f(t)$.



Note: please watch the open/closed circles.

- (b) Write $f(t)$ in terms of unit step functions. Simplify your final answer.

$$\begin{aligned} f(t) &= 0 + (t-1)u_{1,2}(t) + (-t+2)u_{2,3}(t) \\ &= (t-1)u_1(t) - (t-1)u_2(t) + (-t+2)u_2(t) - (-t+2)u_3(t) \\ &= (t-1)u_1(t) + (-2t+3)u_2(t) + (2-t)u_3(t) \\ \boxed{f(t) = (t-1)u_1(t) + (-2t+3)u_2(t) + (2-t)u_3(t)} \end{aligned}$$

Note: $u_{1,2}(t) = u_1(t) - u_2(t)$ ← indicator function

- (c) Find the Laplace transform of $f(t)$.

You can do this either using the definition, or via rule #13 b

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{(t-1)u_1(t)\} + \mathcal{L}\{(-2t+3)u_2(t)\} + \mathcal{L}\{(2-t)u_3(t)\} \xrightarrow{\text{use rule #13 b}} \\ &= e^{-1s} \mathcal{L}\{(t+1)-1\} + e^{-2s} \mathcal{L}\{-2(t+2)+3\} + e^{-3s} \mathcal{L}\{2-(t+3)\} \xrightarrow{\text{simplify}} \\ &= e^{-1s} \mathcal{L}\{t\} + e^{-2s} \mathcal{L}\{-2t-1\} + e^{-3s} \mathcal{L}\{-t-1\} \end{aligned}$$

$$\boxed{F(s) = e^{-1s} \cdot \frac{1}{s^2} + e^{-2s} \cdot \left(-\frac{2}{s^2} - \frac{1}{s}\right) + e^{-3s} \cdot \left(-\frac{1}{s^2} - \frac{1}{s}\right)}$$

Turn over the page for more fun...

2. Find the inverse transformation of $G(s) = \frac{1}{s(s^2 - 2s + 3)}$

roots of the denominator are
 $s=0, s=\frac{1 \pm i\sqrt{2}}{2} \leftarrow$ complex so
 complete the square.

$$G(s) = \frac{1}{s((s-1)^2 + 2)} = \frac{1}{s((s-1)^2 + (\sqrt{2})^2)} = \frac{A}{s} + \frac{Bs + C}{(s-1)^2 + (\sqrt{2})^2} \leftarrow \text{partial fractions}$$

$$A(s^2 - 2s + 3) + (Bs + C)s = 1$$

$$\begin{array}{l} s^0: 3A = 1 \rightarrow A = \frac{1}{3} \\ s^1: -2A + C = 0 \rightarrow C = \frac{2}{3} \\ s^2: A + B = 0 \rightarrow B = -\frac{1}{3} \end{array} \quad \left\{ \begin{array}{l} \\ \\ \end{array} \right. \rightarrow$$

$$\begin{aligned} G(s) &= \frac{1}{3} \cdot \frac{1}{s} + \frac{-\frac{1}{3}s + \frac{2}{3}}{(s-1)^2 + (\sqrt{2})^2} \\ &= \frac{1}{3} \cdot \frac{1}{s} + \frac{\cancel{-\frac{1}{3}(s-1+1) + \frac{2}{3}}}{(s-1)^2 + (\sqrt{2})^2} \rightarrow -\frac{1}{3}(s-1) - \frac{1}{3} + \frac{2}{3} \end{aligned}$$

$$G(s) = \frac{1}{3} \cdot \frac{1}{s} - \frac{1}{3} \cdot \frac{s-1}{(s-1)^2 + (\sqrt{2})^2} + \frac{1}{3} \cdot \frac{1}{(s-1)^2 + (\sqrt{2})^2}$$

this looks like items in the table!

$$g(t) = \frac{1}{3} - \frac{1}{3} e^t \cos(\sqrt{2}t) + \frac{1}{3\sqrt{2}} e^t \sin(\sqrt{2}t)$$

3. Find the inverse transformation of $H(s) = \frac{1}{s(s^2 - 2s + 3)} (e^{-2s} - e^{-4s})$

Using the above ..

$$\begin{aligned} H(s) &= e^{-2s} G(s) - e^{-4s} G(s) \\ h(t) &= u_3(t) \cdot g(t-2) - u_4(t) \cdot g(t-4) \end{aligned}$$

$$\begin{aligned} h_1(t) &= u_3(t) \cdot \left[\frac{1}{3} - \frac{1}{3} e^{t-2} \cos(\sqrt{2}(t-2)) + \frac{1}{3\sqrt{2}} e^{t-2} \sin(\sqrt{2}(t-2)) \right] \\ &\quad - u_4(t) \cdot \left[\frac{1}{3} - \frac{1}{3} e^{t-4} \cos(\sqrt{2}(t-4)) + \frac{1}{3\sqrt{2}} e^{t-4} \sin(\sqrt{2}(t-4)) \right] \end{aligned}$$