

MATH 2340 WARM-UP PROBLEMS

Recall from class on Friday that $\mathcal{L}\{f(t)\}$ is defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

Using integration by parts, we showed that $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$ where $Y(s) = \mathcal{L}\{y(t)\}$.

1. How do you simplify the expression $\mathcal{L}\{y''(t)\}$ in terms of $Y(s)$ where $Y(s) = \mathcal{L}\{y(t)\}$?

$$\begin{aligned}
 \mathcal{L}\{y''(t)\} &= \int_0^\infty e^{-st} y''(t) dt = e^{-st} y'(t) \Big|_0^\infty + s \int_0^\infty e^{-st} y'(t) dt \\
 &= e^{-st} y'(t) \Big|_0^\infty + s(sY(s) - y(0)) \quad \xrightarrow{\text{used } \mathcal{L}\{y'\} = sY(s) - y(0)} \\
 &= 0 - y'(0) + s^2 Y(s) - sy(0) \\
 &= \underline{s^2 Y(s) - sy(0) - y'(0)}
 \end{aligned}$$

2. Find the Laplace transformation of the solution to the IVP

$$y'' - 6y' + 9y = 4, \quad y(0) = 2, \quad y'(0) = 1.$$

Note: ① $\mathcal{L}\{c \cdot f(t)\} = c \mathcal{L}\{f(t)\}$
 ② $\mathcal{L}\{4\} = 4 \mathcal{L}\{1\} = \frac{4}{s}$

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{4\},$$

$$s^2 Y(s) - s \cdot \frac{2}{y(0)} - \frac{1}{y'(0)} - 6 \cdot (sY(s) - 2) + 9Y(s) = \frac{4}{s}$$

collect the $Y(s)$'s

$$Y(s) \cdot (s^2 - 6s + 9) - 2s - 1 + 12 = \frac{4}{s}$$

solve for $Y(s)$:

$$\begin{aligned}
 Y(s) &= \frac{1}{(s^2 - 6s + 9)} \cdot \left(\frac{4}{s} - 11 + 2s \right) \\
 Y(s) &= \boxed{\frac{4 - 11s + 2s^2}{s \cdot (s - 3)(s - 3)}} \quad \xrightarrow{\text{simplify (+ factor denom.)}}
 \end{aligned}$$

Don't turn over the page just yet...

3. Determine what function would give you the following Laplace transforms. Use the table you were given in class on Friday.

$$(a) F(s) = \frac{3}{s-1} = 3 \cdot \frac{1}{s-1}$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s-1}\right\} = 3 \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = \boxed{3 \cdot e^t}$$

by rule #2 with
 $a = 1$.

$$(b) F(s) = \frac{2}{s^2 + 3s - 4} = \frac{2}{(s+4)(s-1)} \Rightarrow \text{split! Use partial fractions!}$$

$$F(s) = \frac{A}{s+4} + \frac{B}{s-1} = \frac{2}{(s+4)(s-1)} \rightarrow A(s-1) + B(s+4) = 2$$

$$\begin{aligned} S^1: A+B &= 0 \\ S^0: -A+4B &= 2 \end{aligned} \quad \left. \begin{array}{l} B = \frac{2}{5}, A = -\frac{2}{5} \end{array} \right\}$$

$$F(s) = -\frac{2}{5} \cdot \frac{1}{s+4} + \frac{2}{5} \cdot \frac{1}{s-1}$$

$$\boxed{f(t) = -\frac{2}{5} e^{-4t} + \frac{2}{5} e^t}$$

using rule #2 w/ $a = -4$ and $a = 1$ respectively.

$$(c) F(s) = \frac{4}{s(s-3)^2} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{(s-3)^2} \leftarrow \text{partial fraction expansion for repeated roots.}$$

$$4 = A(s-3)^2 + Bs(s-3) + Cs$$

$$A = 4/9, B = -4/9, C = 4/3$$

$$F(s) = \frac{4}{9} \cdot \frac{1}{s} - \frac{4}{9} \cdot \frac{1}{s-3} + \frac{4}{3} \cdot \frac{1}{(s-3)^2}$$

rule #11
 $a=3, n=1$

$$\boxed{f(t) = \frac{4}{9} - \frac{4}{9} e^{3t} + \frac{4}{3} t e^{3t}}$$

$$(d) F(s) = \frac{s-1}{s^2 + 4s + 5} = \frac{s-1}{s^2 + 4s + 4 + 1} = \frac{s-1}{(s+2)^2 + 1} = \frac{s+2-2-1}{(s+2)^2 + 1}$$

Warning!
denom. has
complex roots!
Complete the square!

$$= \frac{s+2}{(s+2)^2 + 1^2} - \frac{3}{(s+2)^2 + 1^2}$$

$$\downarrow \text{rule #10} \quad \downarrow \text{rule #9}$$

$\uparrow a=-2, b=1$ $\uparrow a=-2, b=1$

$$\boxed{f(t) = e^{-2t} \cos(1t) - 3e^{-2t} \cdot \sin(1t)}$$