MATH 234 - LECTURE NOTES

REPEATED ROOTS EXAMPLES

EXAMPLE #1

Find the fundamental solution set to the differential equation

$$\vec{\mathbf{u}}' = \begin{bmatrix} -1 & -8 & 4 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \vec{\mathbf{u}} \tag{1}$$

FIND THE SOLUTION

If we assume that the solution $\vec{\mathbf{u}}(t)$ takes the form $\vec{\mathbf{u}}(t) = \vec{\mathbf{v}}e^{\lambda t}$, we get the equation

$$\lambda \vec{\mathbf{v}} e^{\lambda t} = \begin{bmatrix} -1 & -8 & 4 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \vec{\mathbf{v}} e^{\lambda t}.$$

Using the ideas discussed in class, the only way to find a non-zero solution for $\vec{\mathbf{u}}(t)$ is to enforce the condition

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0,$$

where $\mathbf{A}$ is shorthand for the 3×3 matrix (as you prepare for the exam, please make sure you fully understand *why* this condition works). For this problem, we find that $\mathbf{A} - \lambda \mathbf{I}$ is given by

$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} -1-\lambda & -8 & 4 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix}.$$

Finding the Eigenvalues

Using the above calculations, we have that $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ yields

$$(-1-\lambda)\left((2-\lambda)^2 - 1\right) - (-8)\left((2-\lambda) - 1\right) + (4)\left(1 - (2-\lambda)\right) = 0$$

Simplifying the above expression gives:

$$-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = 0 \quad \rightarrow \quad (1-\lambda)^3 = 0$$

This gives only one eigenvalue $\lambda = 1$. Notice that this eigenvalue is a triple root. Thus, the eigenvalue $\lambda_1 = 1$ has an *algebraic multiplicity* of 3.

Finding the Eigenvectors

Now that we know the eigenvalues, we can proceed to find the eigenvectors by solving $(\mathbf{A} - \lambda \mathbf{I})\,\vec{\mathbf{v}} = 0$ for $\vec{\mathbf{v}}$. Since we only have one eigenvalue, we find the equation $(\mathbf{A} - \lambda \mathbf{I})\,\vec{\mathbf{v}} = 0$ becomes

$$\begin{bmatrix} -2 & -8 & 4 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{aligned} -2v_1 - 8v_2 + 4v_3 &= 0, \\ v_1 + v_2 + v_3 &= 0, \\ v_1 + v_2 + v_3 &= 0. \end{aligned}$$

In class, we have been solving these equations by using substitution. Here's let's try something different; let's use elimination.

$$\left[\begin{array}{ccc|c} -2 & -8 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \quad R_2 - R_3 \rightarrow R_3 \quad \left[\begin{array}{ccc|c} -2 & -8 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 + 2R_2 \rightarrow R_2 \quad \left[\begin{array}{ccc|c} -2 & -8 & 4 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rewriting the remaining components as a system, we find

$$\begin{aligned} -2v_1 - 8v_2 + 4v_3 &= 0 \\ -6v_2 + 6v_3 &= 0 \end{aligned}$$

The last equation implies that $v_2 = v_3$. This means, that the first equation becomes

$$-2v_1 - 8v_2 + 4v_2 = 0 \quad \longrightarrow \quad -2v_1 - 4v_2 = 0.$$

This last condition implies that $v_1 = -2v_2$. Thus, we have $\vec{\mathbf{v}}$ given by

$$\vec{\mathbf{v}} = \begin{bmatrix} -2v_2 \\ v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

For simplicity, we will choose $v_2 = 1$.

This means that for the eigenvalue $\lambda = 1$, we only found one eigenvector

$$\vec{\mathbf{v}} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

This means that the geometric multiplicity of the eigenvector is 1. Thus, we only get one fundamental solution from the eigenvectors, and it is given by

$$\vec{\mathbf{u}}^{(1)}(t) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} e^t.$$

Finding the second fundamental solution.

Since we only found one fundamental solution, we will guess the form for the second fundamental solution based on the form $\vec{\mathbf{u}}^{(2)} = tu^{(1)} + \vec{\mathbf{w}}e^{\lambda t}$, where $\vec{\mathbf{w}}$ is some new vector that we must find. This gives

$$\vec{\mathbf{u}}^{(2)} = te^t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \vec{\mathbf{w}}e^t.$$

Since we want $\vec{\mathbf{u}}^{(2)}$ to be a solution, it should satisfy the original differential equation. Plugging in this guess into Equation (1), we find

$$e^t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + te^t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \vec{\mathbf{w}}e^t = \begin{bmatrix} -1 & -8 & 4 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \left(te^t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \vec{\mathbf{w}}e^t \right)$$

The above equation simply reduces to the following equation (verify this for yourself!):

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & -8 & 4 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \vec{\mathbf{w}}$$

Note: this is the same form as we expect: $\vec{\mathbf{v}}^{(1)} = (\mathbf{A} - \lambda \mathbf{I})\, \vec{\mathbf{w}}$ In order to solve for $\vec{\mathbf{w}}$, it might be helpful to write $\vec{\mathbf{w}}$ in component form as follows:

$$\begin{bmatrix} -2 & -8 & 4 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \rightarrow \quad \begin{aligned} -2w_1 - 8w_2 + 4w_3 &= -2, \\ w_1 + w_2 + w_3 &= 1, \\ w_1 + w_2 + w_3 &= 1. \end{aligned}$$

Following the same pattern as before, we have

$$\left[\begin{array}{ccc|c} -2 & -8 & 4 & -2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] \quad R_2 - R_3 \to R_3 \quad \left[\begin{array}{ccc|c} -2 & -8 & 4 & -2 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 + 2R_2 \to R_2 \quad \left[\begin{array}{ccc|c} -2 & -8 & 4 & -2 \\ 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rewriting the remaining components as a system, we find

$$\begin{aligned} -2w_1 - 8w_2 + 4w_3 &= -2 \\ -6w_2 + 6w_3 &= 0 \end{aligned}$$

The last equation implies that $w_2 = w_3$. This means, that the first equation becomes

$$-2w_1 - 8w_2 + 4w_2 = -2 \quad \longrightarrow \quad -2w_1 - 4w_2 = -2.$$

This last condition implies that $w_1 = 1 - 2w_2$. Thus, we have $\vec{\mathbf{w}}$ given by

$$\vec{\mathbf{w}} = \begin{bmatrix} 1 - 2w_2 \\ w_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w_2 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

This means that the general solution for $\vec{\mathbf{u}}^{(2)}$ is given by

$$\vec{\mathbf{u}}^{(2)} = te^t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + e^t \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w_2 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right).$$

Since the term

$$w_2 e^t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is simply a constant multiple of the first fundamental solution, we can choose to ignore this term by selecting $w_2 = 0$.

Thus, so far we have the following:

$$\vec{\mathbf{u}}^{(1)} = e^t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{\mathbf{u}}^{(2)} = te^t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

Finding the third fundamental solution.

So far, we only have two of the needed fundamental solutions. We need to find the third fundamental solution in order to obtain the general solution. Let's assume that we have no idea what we should really guess for $\vec{\mathbf{u}}^{(3)}$; so let's try something generic. In other words, let's try

$$\vec{\mathbf{u}}_3 = t^2 e^{\lambda t} \vec{\mathbf{a}} + te^{\lambda t} \vec{\mathbf{b}} + e^{\lambda t} \vec{\mathbf{c}},$$

where $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ are all unknown terms. Notice that this guess incorporates a t^2 term as well.

Since we want $\vec{\mathbf{u}}^{(3)}$ to be a solution, it should satisfy the original differential equation. Plugging in this guess into Equation (1), we find

$$2te^{\lambda t}\vec{\mathbf{a}} + \lambda t^2 e^{\lambda t}\vec{\mathbf{a}} + e^{\lambda t}\vec{\mathbf{b}} + \lambda te^{\lambda t}\vec{\mathbf{b}} + \lambda e^{\lambda t}\vec{\mathbf{c}} = \begin{bmatrix} -1 & -8 & 4 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \left(t^2 e^{\lambda t}\vec{\mathbf{a}} + te^{\lambda t}\vec{\mathbf{b}} + e^{\lambda t}\vec{\mathbf{c}} \right)$$

If we collect like powers of t, we end up with three basic equations:

t^2: $\lambda \vec{\mathbf{a}} = \mathbf{A}\vec{\mathbf{a}}$
t^1: $2\vec{\mathbf{a}} + \lambda \vec{\mathbf{b}} = \mathbf{A}\vec{\mathbf{b}}$
t^0: $\vec{\mathbf{b}} + \lambda \vec{\mathbf{c}} = \mathbf{A}\vec{\mathbf{c}}$

The above system of equations can be solved. For example, the first equation for $\vec{\mathbf{a}}$ is simply the same condition as finding the eigenvalues of $\mathbf{A}$. Thus,

$$\vec{\mathbf{a}} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

recalling that $\lambda = 1$. Once we know $\vec{\mathbf{a}}$, we can find $\vec{\mathbf{b}}$ which gives the equation

$$(\mathbf{A} - \mathbf{I})\,\vec{\mathbf{b}} = 2\vec{\mathbf{a}}$$

Solving the above gives

$$\vec{\mathbf{b}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Finally, solving the last equation gives

$$\vec{\mathbf{c}} = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}.$$

Thus, we have the final fundamental solution

$$\vec{\mathbf{u}}^{(3)} = t^2 e^t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + te^t \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix} e^t.$$

Some final comments...

Note: with the above solution, we could pull out a factor of 2 to get

$$\vec{\mathbf{u}}^{(3)} = 2\left(\frac{1}{2}t^2 e^t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + te^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{6} \\ 0 \\ \frac{1}{6} \end{bmatrix} e^t\right).$$

Why would we want to do this? Well, notice that we have the solution in the following form:

$$\vec{\mathbf{u}}^{(3)} = 2\left(\frac{1}{2}t^2 e^t \vec{\mathbf{v}} + te^t \vec{\mathbf{w}}^{(1)} + \vec{\mathbf{w}}^{(2)} e^t\right)$$

where $w^{(1)}$ is the vector we found for the second fundamental solution and $\vec{\mathbf{w}}^{(2)}$ is given by

$$\vec{\mathbf{w}}^{(2)} = \begin{bmatrix} -\frac{1}{6} \\ 0 \\ \frac{1}{6} \end{bmatrix}$$

It turns out that $\vec{\mathbf{w}}^{(2)}$ satisfies $(\mathbf{A} - \lambda\mathbf{I})\,\vec{\mathbf{w}}^{(2)} = \vec{\mathbf{w}}^{(1)}$. In fact, we can generalize this idea so that if the eigenvalue has a large enough "defect" (the geometric multiplicity γ is less than the algebraic multiplicity μ where $\mu - \gamma = k$), we can find the following:

$$\vec{\mathbf{u}}^{(j)} = \left(\frac{1}{(j-1)!}\vec{\mathbf{v}}t^{j-1} + \frac{1}{(j-2)!}\vec{\mathbf{w}}^{(1)}t^{j-2} + \ldots + \vec{\mathbf{w}}^{(j-1)}\right)e^{\lambda t}, \quad j = \gamma, \ldots, \mu$$

where the $\vec{\mathbf{w}}^{(j)}$'s can be found through the following hierarchy:

$$\begin{aligned}
(\mathbf{A} - \lambda\mathbf{I})\,\vec{\mathbf{w}}^{(1)} &= \vec{\mathbf{v}} \\
(\mathbf{A} - \lambda\mathbf{I})\,\vec{\mathbf{w}}^{(2)} &= \vec{\mathbf{w}}^{(1)} \\
&\vdots \\
(\mathbf{A} - \lambda\mathbf{I})\,\vec{\mathbf{w}}^{(k)} &= \vec{\mathbf{w}}^{(k-1)}
\end{aligned}$$