(*) The case of the Repeated Roots

<u>Ex</u>: Find the general solution to

$$\vec{u}' = A\vec{u} \quad \text{where} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$

evals: $\det(A - \lambda I) = 0$

$$(1-\lambda)^2 - 0 = 0$$

$\underline{\lambda = 1 \text{ repeated!}}$

evects: $\lambda_1 = 1$

$$\begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0v_1 + 1v_2 = 0 \Rightarrow v_2 = 0$ $v_1 =$ whatever it wants!

$\lambda_1 = 1$ $\vec{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ only 1 evector!

How do we find 2^{nd} fund sol?

Guess: $\vec{u} = \vec{v}te^{\lambda t}$? $\vec{u}' = \vec{v}e^{\lambda t} + \vec{v}\lambda t e^{\lambda t}$

$$\vec{v}\cancel{e^{\lambda t}}(1 + \lambda t) = At\cancel{e^{\lambda t}}\vec{v}$$

$\vec{v}(1 + \lambda t) = At\vec{v}$ but if you equate powers of t

$$\left.\begin{aligned} \vec{v} &= 0 \\ \lambda\vec{v} &= A\vec{v} \end{aligned}\right\} \text{ Boo! Doesn't work!}$$

What do we do?

Go back to the beginning:

$$\vec{u}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{u}$$

$u_1' = u_1 + u_2$

$u_2' = u_2$ ← separate!

$u_2' = u_2 \rightarrow \boxed{u_2 = c_1 e^t} \rightarrow$ plug into 1st ode.

$$u_1' = u_1 + c_1 e^t$$

$$u_1' - u_1 = c_1 e^t$$ → linear 1st order → int. factor!

$$\mu(t) = e^{\int -dt} = e^{-t}$$

$$\frac{d}{dt}\left(u_1 e^{-t}\right) = c_1 \cancel{e^t} \cancel{e^{-t}}$$

const. of integration

$$u_1 e^{-t} = c_1 t + c_2$$

$$\boxed{u_1 = c_1 t e^t + c_2 e^t}$$

$$\vec{u} = c_1 \begin{bmatrix} te^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

$$\vec{u} = c_1 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} te^t \right) + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$$

what's this guy?

↓

some other vector

$\vec{v}^{(1)}$ our 1st e-vect!

$\lambda^2 + 4\lambda + 4$

But this tells us "how" to guess.

$$\vec{u}^{(1)} = \vec{v}^{(1)} e^{\lambda_1 t}, \quad \vec{u}^{(2)} = \vec{v}^{(1)} t e^{\lambda_1 t} + \vec{v}^{(2)} e^{\lambda_1 t}$$

If we guess $\vec{u}^{(2)}$ of this form, we can find $\vec{v}^{(2)}$

Since we already know the ans for this problem, try something different.

(*) $\vec{u}' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \vec{u}$

$A - \lambda I = \begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\Rightarrow (3-\lambda)(1-\lambda) + 1 = 0$

$3 - 4\lambda + \lambda^2 + 1 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$\lambda = 2$ repeated.

$\lambda = 2 \qquad \vec{v}^{(1)}\,?$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to v_1 - v_2 = 0 \quad \boxed{v_1 = v_2} \quad \vec{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{u}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

Find 2nd fund sol. $\to$ $\vec{u}^{(2)} = \vec{v}^{(1)} t e^{\lambda t} + \vec{v}^{(2)} e^{\lambda t}$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{2t} + \vec{v}^{(2)} e^{2t}$$

$$\vec{u}^{(2)\prime} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} t e^{2t} + 2\vec{v}^{(2)} e^{2t}$$

plug in

$$\vec{v}^{(1)}e^{2t} + 2\vec{v}^{(1)}te^{2t} + 2\vec{v}^{(2)}e^{2t} = A\vec{v}^{(1)}te^{2t} + A\vec{v}^{(2)}e^{2t}$$

$e^{2t}t^1$: $2\vec{v}^{(1)} = A\vec{v}^{(1)} \rightarrow$ it's true! if $\lambda = 2$

$A\vec{v} = 2\vec{v}$ ✓

$e^{2t}t^0$: $\vec{v}^{(1)} + 2\vec{v}^{(2)} = A\vec{v}^{(2)}$

$$\vec{v}^{(1)} = (\underbrace{A - 2I}_{}\,)\vec{v}^{(2)}$$

already know
$\det(A-2I)=0$
$\Rightarrow \infty$ # of sols.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$v_1 - v_2 = 1$
$v_1 - v_2 = 1$

$v_1 = 1 + v_2$

$$\vec{v}^{(2)} = \begin{bmatrix} 1+k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{u}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}te^{2t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{2t} + \underbrace{k\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t}}_{}$$

already $\vec{u}^{(1)}(t)$
so we could ignore.

$$\boxed{\vec{u}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}te^{2t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{2t}}$$