

Non Homogeneous Systems

$$\vec{u}' = A\vec{u} + \vec{g}(t)$$

The general solution to the non-homog. problem consists of a homogeneous (complementary) solution and a particular solution.

$$\vec{u}(t) = \vec{u}_h(t) + \vec{u}_p(t)$$

$$\vec{u}_h(t) \text{ solves } \vec{u}' = A\vec{u} \quad \rightarrow \quad \vec{u}_h = \Phi(t)\vec{u}_0 \quad \text{or} \quad \vec{u}_h = \Psi(t)\vec{c}$$

$$\vec{u}_p(t) \text{ solves } \vec{u}' = A\vec{u} + \vec{g}(t)$$

So, we know how to find $\vec{u}_h(t)$, but what about \vec{u}_p ?

- * Method of Undetermined Coeffs. ← this turns out to be a mess 9/10
 - * Variation of Parameters
 - * Laplace Transforms
- we will focus on these two.

Variation of Parameters

Guess $\vec{u}_p = \Psi(t)\vec{v}(t)$

where we replaced the \vec{c}_i with an unknown function $\vec{v}(t)$. Let the parameters vary!!

plug into the full equation:

$$\vec{u}' = A\vec{u} + \vec{g}(t) \quad \rightarrow \quad \underbrace{\Psi'\vec{v} + \Psi\vec{v}'}_{\text{product rule}} = A\Psi\vec{v} + \vec{g}(t)$$

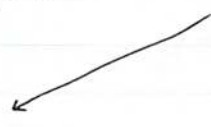
Now, here's what's interesting...

$\Psi' = A\Psi \rightarrow$ the fundamental matrix satisfies the differential equation $\vec{u}' = A\vec{u}$

This gives

$$\Psi' \vec{v} + \Psi \vec{v}' = A\Psi \vec{v} + \vec{g}(t)$$

$$\Psi \vec{v}' = \vec{g}(t) \rightarrow \vec{v}'(t) = \Psi^{-1}(t) \vec{g}(t)$$



$$\vec{v} = \int \Psi^{-1}(t) \vec{g}(t) dt$$

So, we have $\vec{u}_p(t) = \Psi(t) \int \Psi^{-1}(t) \vec{g}(t) dt$

and more generally

$$\vec{u} = \vec{u}_h + \vec{u}_p$$

$$\vec{u} = \Psi(t) \vec{c} + \Psi(t) \int \Psi^{-1}(t) \vec{g}(t) dt \quad \text{must solve for } \vec{c}.$$

Or, you can show

$$\vec{u} = \Phi(t) \vec{u}_0 + \Phi(t) \int_0^t \Phi^{-1}(t) \vec{g}(t) dt \quad \text{no solving for } \vec{c}.$$

So, variation of parameters says that the general solution to

$$\vec{u}' = A\vec{u} + \vec{g}(t) \quad \text{is}$$

$$\vec{u}(t) = \Psi(t)\vec{c} + \Psi(t) \int \Psi^{-1}(t)\vec{g}(t) dt$$

or

$$\vec{u}(t) = \Phi(t)\vec{u}_0 + \Phi(t) \int_0^t \Phi^{-1}(t)\vec{g}(t) dt$$

advantage: no solving for \vec{c} !

Alternatively, you can use Laplace Transforms! However, this method is best suited for const. coeff. problems.

Consider the IVP

$$\vec{u}'(t) = A\vec{u}(t) + \vec{g}(t), \quad \vec{u}(0) = \vec{u}_0$$

Taking the Laplace transform of both sides, we have

$$\mathcal{L}\{\vec{u}'(t)\} = \mathcal{L}\{A\vec{u}(t) + \vec{g}(t)\}$$

$$\text{let } \vec{U}(s) = \mathcal{L}\{\vec{u}(t)\} \\ \vec{G}(s) = \mathcal{L}\{\vec{g}(t)\}$$

$$\hookrightarrow s\vec{U}(s) - \vec{u}_0(t) = A\vec{U}(s) + \vec{G}(s)$$

Solving for $\vec{U}(s)$, we have ...

$$(s\mathbf{I} - \mathbf{A}) \vec{U}(s) = \vec{u}_0 + \vec{G}(s)$$

$$\downarrow$$

$$\vec{U}(s) = (s\mathbf{I} - \mathbf{A})^{-1} [\vec{u}_0 + \vec{G}(s)] = (s\mathbf{I} - \mathbf{A})^{-1} \vec{u}_0 + (s\mathbf{I} - \mathbf{A})^{-1} \vec{G}(s)$$

To find $\vec{u}(t)$, we simply take the inverse Laplace transform to find...

$$\vec{u}(t) = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1} \vec{u}_0\} + \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1} \vec{G}(s)\}$$

Alternatively, we can rewrite it as

$$\vec{u}(t) = \underbrace{\mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}}_{\text{note: this is the fundamental matrix } \Phi(t)!} \vec{u}_0 + \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1} \vec{G}(s)\}$$

note: this is
the fundamental
matrix $\Phi(t)$!

$$\mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\} = \Phi(t)$$