Non Homogeneous Systems

$$\vec{u}' = A\vec{u} + \vec{g}(t)$$

The general solution to the non-homog. problem consists of a homogeneous (complementary) solution and a particular solution.

$$\vec{u}(t) = \vec{u}_h(t) + \vec{u}_p(t)$$

$\vec{u}_h(t)$ solves $\vec{u}' = A\vec{u}$ $\longrightarrow$ $\vec{u}_h = \Phi(t)\vec{u}_0$ or $\vec{u}_h = \Psi(t)\vec{c}$

$\vec{u}_p(t)$ solves $\vec{u}' = A\vec{u} + \vec{g}(t)$

So, we know how to find $\vec{u}_h(t)$, but what about $\vec{u}_p$?

- Method of Undetermined Coeffs. ← this turns out to be a mess 9/10
- Variation of Parameters
- Laplace Transforms

we will focus on these two.

Variation of Parameters

Guess $\vec{u}_p = \Psi(t)\vec{v}(t)$ where we replaced the $\vec{c}$ with an unknown function $\vec{v}(t)$. Let the parameters vary!!

plug into the full equation:

$$\vec{u}' = A\vec{u} + \vec{g}(t) \longrightarrow \underbrace{\Psi'\vec{v} + \Psi\vec{v}'}_{\text{product rule}} = A\Psi\vec{v} + \vec{g}(t)$$

Now, here's what's interesting...

$\Psi' = A\Psi \rightarrow$ the fundamental matrix satisfies the differential equation $\vec{u}' = A\vec{u}$

This gives

$$\Psi'\vec{v} + \Psi\vec{v}' = A\cancel{\Psi}\vec{v} + \vec{g}(t)$$

$$\Psi\vec{v}' = \vec{g}(t) \longrightarrow \vec{v}'(t) = \Psi\ (t)\vec{g}(t)$$

$$\vec{v} = \int \Psi^{-1}(t)\vec{g}(t)\,dt$$

So, we have $\vec{u}_p(t) = \Psi(t)\int \Psi^{-1}(t)\vec{g}(t)\,dt$

and more generally

$$\vec{u} = \vec{u}_h + \vec{u}_p$$

$\vec{u} = \Psi(t)\vec{c} + \Psi(t)\int \Psi^{-1}(t)\vec{g}(t)\,dt$ must solve for $\vec{c}$.

Or, you can show

$\vec{u} = \Phi(t)\vec{u}_0 + \Phi(t)\int_0^t \Phi^{-1}(t)\vec{g}(t)\,dt$ no solving for $\vec{c}$.

So, variation of parameters says that the general solution to

$$\vec{u}' = A\vec{u} + \vec{g}(t) \quad \text{is}$$

$$\vec{u}(t) = \Psi(t)\vec{c} + \Psi(t)\int \Psi^{-1}(t)\vec{g}(t)\,dt$$

or

$$\vec{u}(t) = \Phi(t)\vec{u}_0 + \Phi(t)\int_0^t \Phi^{-1}(t)\vec{g}(t)\,dt$$

advantage: no solving for $\vec{c}$!

Alternatively, you can use Laplace Transforms! However, this method is best suited for const. coeff. problems.

Consider the IVP

$$\vec{u}'(t) = A\vec{u}(t) + \vec{g}(t), \qquad \vec{u}(0) = \vec{u}_0$$

Taking the Laplace transform of both sides, we have

$$\mathcal{L}\{\vec{u}'(t)\} = \mathcal{L}\{A\vec{u}(t) + \vec{g}(t)\}$$

let $\vec{U}(s) = \mathcal{L}\{\vec{u}(t)\}$
$\vec{G}(s) = \mathcal{L}\{\vec{g}(t)\}$

$$\hookrightarrow \quad s\vec{U}(s) - \vec{u}_0(t) = A\vec{U}(s) + \vec{G}(s)$$

Solving for $U(s)$, we have ...

$$(sI - A)\vec{U}(s) = \vec{u}_0 + \vec{G}(s)$$

$$\downarrow$$

$$\vec{U}(s) = (sI - A)^{-1}\left[\vec{u}_0 + \vec{G}(s)\right] = (sI-A)^{-1}\vec{u}_0 + (sI-A)^{-1}\vec{G}(s)$$

To find $\vec{u}(t)$, we simply take the inverse Laplace transform to find...

$$\vec{u}(t) = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\vec{u}_0\right\} + \mathcal{L}^{-1}\left\{(sI-A)^{-1}\vec{G}(s)\right\}$$

Alternatively, we can rewrite it as

$$\vec{u}(t) = \underbrace{\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}\vec{u}_0 + \mathcal{L}^{-1}\left\{(sI-A)^{-1}\vec{G}(s)\right\}$$

note: this is the fundamental matrix $\Phi(t)$!

$$\boxed{\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \Phi(t)}$$