12. (15 points) Find the solution to the initial value problem given by

$$\vec{u}' = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \vec{u} + \begin{bmatrix} e^{4t} \\ e^{2t} \end{bmatrix}, \quad \vec{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using variation of parameters, $\vec{u}(t) = \Phi(t)\vec{u}_0 + \Phi(t)\int_0^t \Phi^{-1}(t)\vec{g}(t)\,dt$

* Step 1: Find the fundamental matrix $\Phi(t)$ (find fund. sols to $\vec{u}' = A\vec{u}$)

$$\begin{bmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{bmatrix} \vec{v} = \vec{0}$$

$$\det \begin{bmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{bmatrix} = 0 \rightarrow (2-\lambda)(4-\lambda) + 1 = 0$$

$$8 - 6\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - 6\lambda + 9 = 0 \rightarrow \underline{\lambda = 3} \text{ repeated.}$$

$\boxed{\lambda = 3}$

$$\begin{bmatrix} 2-3 & -1 \\ 1 & 4-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \left.\begin{matrix} -v_1 - v_2 = 0 \\ v_1 + v_2 = 0 \end{matrix}\right\} \rightarrow v_1 = -v_2 \rightarrow \vec{v}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So, one fundamental solution is given by

$$\underline{\vec{u}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}} = \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix}$$

The second is found by assuming $\vec{u}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{3t} + \vec{v}^{(2)} e^{3t}$

This results in the equation:

$$(A - 3I)\vec{v}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \begin{matrix} v_1 + v_2 = -1 \\ \underline{v_1 = -1 - v_2} \end{matrix} \quad \vec{v}^{(2)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\underline{\vec{u}^{(2)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{3t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{3t}} = \begin{bmatrix} te^{3t} - e^{3t} \\ -te^{3t} \end{bmatrix}$$

$$\text{So, } \Psi(t) = \begin{bmatrix} e^{3t} & te^{3t} - e^{3t} \\ -e^{3t} & -te^{3t} \end{bmatrix} \rightarrow \Psi(0) = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \rightarrow \Psi^{-1}(0) = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\Phi(t) = \Psi(t)\Psi^{-1}(0) = \begin{bmatrix} -te^{3t} + e^{3t} & -te^{3t} \\ + te^{3t} & + e^{3t} + te^{3t} \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} -te^{3t} + e^{3t} & -te^{3t} \\ te^{3t} & e^{3t} + te^{3t} \end{bmatrix}$$

* <u>Step 2</u>: Find $\Phi^{-1}(t)$

$$\Phi^{-1}(t) = \frac{1}{\det(\Phi(t))} \cdot \begin{bmatrix} e^{3t} + te^{3t} & te^{3t} \\ -te^{3t} & -te^{3t} + e^{3t} \end{bmatrix}$$

$$\det\left(\Phi(t)\right) = e^{6t} - t^2 e^{6t} + t^2 e^{6t} = e^{6t}$$

$$\phi^{-1}(t) = e^{-6t} \begin{bmatrix} e^{3t} + te^{3t} & te^{3t} \\ -te^{3t} & e^{3t} - te^{3t} \end{bmatrix}$$

$$\Phi^{-1}(t) = \begin{bmatrix} e^{-3t} + te^{-3t} & te^{-3t} \\ -te^{-3t} & e^{-3t} - te^{-3t} \end{bmatrix}$$

* <u>Step 3</u>: Find $\Phi^{-1}(t)\,\vec{g}(t)$

$$\phi^{-1}(t)\vec{g}(t) = \begin{bmatrix} e^{-3t} + te^{-3t} & te^{-3t} \\ -te^{-3t} & e^{-3t}(1-t) \end{bmatrix} \begin{bmatrix} e^{4t} \\ e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{t}(1+t) + te^{-t} \\ -te^{t} + e^{-t}(1-t) \end{bmatrix} = \begin{bmatrix} e^{t} + te^{t} + te^{-t} \\ e^{-t} - te^{-t} - te^{t} \end{bmatrix}$$

* Step 4 : $\int_0^t \Phi^{-1}(t)\,\vec{g}(t)\,dt$

$$= \begin{bmatrix} \int_0^t e^t + te^t + te^{-t}\,dt \\ \\ \int_0^t e^{-t} - te^{-t} - te^t\,dt \end{bmatrix} = \begin{bmatrix} 1 - e^{-t} - te^{-t} + te^t \\ \\ -1 + e^t(1-t) + e^{-t}t \end{bmatrix}$$

* Step 5 $\Phi(t) \cdot \int_0^t \Phi^{-1}(t)\,\vec{g}(t)\,dt$

$$= \begin{bmatrix} e^{3t}(1-t) & -te^{3t} \\ \\ te^{3t} & e^{3t}(1+t) \end{bmatrix} \cdot \begin{bmatrix} 1 - e^{-t}(1+t) + te^t \\ \\ -1 + e^t(1-t) + te^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{3t}(1-\cancel{t}) - e^{2t}(1-\cancel{t^2}) + te^{4t}(\cancel{1-t}) + \cancel{te^{3t}} - te^{4t}(\cancel{1-t}) - \cancel{t^2e^{2t}} \\ \\ \cancel{te^{3t}} - te^{2t}(\cancel{1+t}) + \cancel{t^2e^{4t}} - e^{3t}(1+\cancel{t}) + e^{4t}(1-\cancel{t^2}) + te^{2t}(\cancel{1+t}) \end{bmatrix}$$

$$= \begin{bmatrix} e^{3t} - e^{2t} \\ e^{4t} - e^{3t} \end{bmatrix}$$

* Step 6 $\vec{u} = \Phi(t)\,\vec{u}_0 + \Phi(t)\int_0^t \Phi^{-1}(t)\,\vec{g}(t)\,dt$

$$\boxed{\vec{u}(t) = \begin{bmatrix} e^{3t} - e^{2t} \\ e^{4t} - e^{3t} \end{bmatrix}}$$