## The Fundamental Matrix

Consider the system

$$\vec{u}' = A\vec{u} \qquad \text{where} \quad A = \text{constant } n\times n \text{ matrix}$$
$$\vec{u} = n \text{ dimensional vector.}$$

The general solution is given by

$$\vec{u}(t) = c_1\vec{u}^{(1)}(t) + c_2\vec{u}^{(2)}(t) + \dots + c_n\vec{u}^{(n)}(t)$$

where $\vec{u}^{(j)}$ is the $j$-th fundamental solution.

Alternatively, this can be written in Matrix form as

$$\vec{u}(t) = \left[\vec{u}^{(1)}(t) \;\vdots\; \vec{u}^{(2)}(t) \;\vdots\; \cdots \;\vdots\; \vec{u}^{(n)}(t)\right]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Ex:

$$\vec{u}(t) = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{2t} + c_2\begin{bmatrix}1\\0\end{bmatrix}e^{-3t}$$

this can be rewritten as:

$$\Rightarrow \begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Why care? Try solving the IVP $\vec{u}' = A\vec{u}$, $\vec{u}(0) = \vec{u}_0$

If you know that the general sol to $\vec{u}' = A\vec{u}$ is given by

$$\vec{u}(t) = \left[\vec{u}^{(1)} \;\vdots\; \vec{u}^{(2)} \;\vdots\; \cdots \;\vdots\; \vec{u}^{(n)}\right]\vec{c} \qquad \text{where} \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

then it's just linear algebra to solve for the constants $\vec{c}$.

Plug in $t=0$ into $\vec{u}(t) = \left[\vec{u}^{(1)} \,\vdots\, \cdots \,\vdots\, \vec{u}^{(n)}\right]\vec{c}$ and set equal to $\vec{u}_0$

$$\underset{\text{given}}{\vec{u}_0} = \left[\underbrace{\vec{u}^{(1)}(0) \,\vdots\, \vec{u}^{(2)}(0) \,\vdots\, \cdots \,\vdots\, \vec{u}^{(n)}}_{\text{fundamental sols evaluated @ } t=0}\right] \underset{\text{unknown}}{\vec{c}}$$

Solving for $\vec{c}$, we have

$$\vec{c} = \left[\vec{u}^{(1)}(0) \,\vdots\, \vec{u}^{(2)}(0) \,\vdots\, \cdots \,\vdots\, \vec{u}^{(n)}(0)\right]^{-1} \vec{u}_0$$

For simplicity of notation, let $\Psi(t) = \left[\vec{u}^{(1)}(t) \,\vdots\, \vec{u}^{(2)}(t) \,\vdots\, \cdots \,\vdots\, \vec{u}^{(n)}(t)\right]$,

then we have $\vec{c} = \Psi^{-1}(0)\,\vec{u}_0$ and thus

$$\vec{u}(t) = \Psi(t)\vec{c} \longrightarrow \vec{u}(t) = \Psi(t)\Psi^{-1}(0)\,\vec{u}_0$$

Question: How do you know $\Psi^{-1}(0)$ exists?

Ans: Think Wronskian!

We call $\Phi(t) = \Psi(t)\Psi^{-1}(t_0)$ the <u>fundamental matrix</u>.

To find the fundamental matrix, for $\vec{u}' = A\vec{u}$

1. Find $n$ linearly independ. fund. sols. $\vec{u}^{(1)}(t), \ldots, \vec{u}^{(n)}(t)$
2. Find $\Psi(t) = \left[\vec{u}^{(1)}(t) \,\vdots\, \vec{u}^{(2)}(t) \,\vdots\, \cdots\, \vec{u}^{(n)}(t)\right]$
3. Find $\Psi(0)$ and $\left[\Psi(0)\right]^{-1}$
4. $\Phi(t) = \Psi(t)\Psi^{-1}(0)$

Then, the general solution can be expressed in terms of $\Phi(t)$ as

$$\boxed{\vec{u}(t) = \Phi(t)\,\vec{u}_0}$$

Question: Prove that $\Phi'(t) = A\,\Phi(t)$

In other words, show that $\Phi(t)$ (which is a matrix)

satisfies the differential equation $\vec{u}' = A\vec{u}$.

Example:

Find the fundamental matrix $\Psi(t)$ and $\Phi(t)$ for the differential equation

$$\vec{u}' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \vec{u}$$

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From the repeated roots notes, we know that the fundamental solutions are given by:

$$\vec{u}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} \qquad \vec{u}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} = \begin{bmatrix} te^{2t} + e^{2t} \\ te^{2t} \end{bmatrix}$$

$$\Rightarrow \boxed{\Psi(t) = \left[\, \vec{u}^{(1)}(t) \;\vdots\; \vec{u}^{(2)}(t) \,\right] = \begin{bmatrix} e^{2t} & \vdots & te^{2t} + e^{2t} \\ e^{2t} & \vdots & te^{2t} \end{bmatrix}}$$

Then,

$$\Psi(0) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \Psi^{-1}(0) = \frac{1}{-1}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\therefore \quad \Phi(t) = \Psi(t)\Psi^{-1}(0)$$

$$= \begin{bmatrix} e^{2t} & te^{2t} + e^{2t} \\ e^{2t} & te^{2t} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} te^{2t} + e^{2t} & e^{2t} - (te^{2t} + e^{2t}) \\ te^{2t} & e^{2t} - te^{2t} \end{bmatrix}$$

$$\boxed{\Phi(t) = \begin{bmatrix} te^{2t} + e^{2t} & -te^{2t} \\ te^{2t} & e^{2t} - te^{-2t} \end{bmatrix}}$$

this is a 2×2 matrix