Systems: 3×3 Unique Roots

Monday, June 3, 2013 9:27 AM

Example:

Find the general solution to the ODE

$$\vec{u}' = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \vec{u}$$

Solution:

Since the ODE is alinear, first order system of const. coeff. homog. ODEs,

① Guess $\vec{u} = \vec{v} e^{\lambda t}$ goal: find $\vec{v}$ (const. vector) and λ (const. scalar)

② Plug in guess into ODE: side calc: $\vec{u}' = \lambda \vec{v} e^{\lambda t}$

$$\lambda \vec{v} \cancel{e^{\lambda t}} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \vec{v} \cancel{e^{\lambda t}}$$

cancel $e^{\lambda t}$ since $e^{\lambda t} \neq 0$.

rearrange terms to the form $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 1-\lambda & 2 & 1 \\ 2 & -1-\lambda & 3 \\ 2 & 1 & 1-\lambda \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

from linear algebra, the only way that this equation will have a non-zero sol is if $\det(A - \lambda I) = 0$.

③ Solve $\det(A - \lambda I) = 0$ for λ.

recall notation $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A - \lambda I) = (1-\lambda)\begin{vmatrix} -1-\lambda & 3 \\ 1 & 1-\lambda \end{vmatrix} - (2)\begin{vmatrix} 2 & 3 \\ 2 & 1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 2 & -1-\lambda \\ 2 & 1 \end{vmatrix}$$

$$= (1-\lambda)\left((-1+\lambda^2) - 3\right) - 2\left(2(1-\lambda) - 6\right) + 1\left(2 - 2(-1-\lambda)\right)$$

$$= -\lambda^3 + \lambda^2 - 10\lambda + 8$$

$\det(A-\lambda I)=0 \implies -\lambda^3+\lambda^2-10\lambda+8=0$

Solve any way you want. A quick graph yields:

$\boxed{\text{eigenvalues are } \lambda=-2,\ \lambda=-1, \text{ and } \lambda=4}$

④ For each λ, find the corresponding $\vec{v}$.

a) plug λ into $(A-\lambda I)\vec{v}=0$

b) solve for $\vec{v}$.

$\boxed{\text{For } \lambda=-2}$

$$(A-\lambda I)\vec{v}=0 \implies \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}\vec{v}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can try elimination in order to solve for $\vec{v}$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right] \quad R_2-R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

try to get a zero here

try to get a zero here

$2R_1-3R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$3v_1+2v_2+1v_3=0$

$v_2+7v_3=0 \to v_2=+7v_3$

Since $v_2=7v_3$, we have $3v_1+2(7v_3)+v_3=0$ or...

$3v_1+15v_3=0 \qquad v_1=-5v_3$

So, $\vec{v}$ is any vector of the form

$$\vec{v}=\begin{bmatrix} -5v_3 \\ 7v_3 \\ 1v_3 \end{bmatrix}=\begin{bmatrix} -5 \\ 7 \\ 1 \end{bmatrix}v_3$$

We can proceed to find the following

for $\lambda = -1$, $\vec{v} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$ for $\lambda = 4$ $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(5) The general solution:

$$\boxed{\vec{u} = c_1 \begin{bmatrix} -5 \\ 7 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}}$$

vector form of solution.