MATH 234 (02)
FORCED MECHANICAL VIBRATIONS
(PRELIMINARY DRAFT)

THE FORCED PROBLEM - THE NONHOMOGENEOUS PROBLEM

Let's go back to the set-up for the mass-spring-damper system from the previous lecture, but now, let's include an external forcing term

$$F_e(t) = F_0 \cos \omega t.$$

Here F_0 and Ω are constants. This forcing represents a periodic moving up and down of the base of the spring system with constant amplitude F_0 and frequency Ω. Thus the differential equation we're looking at is

$$mx'' + \gamma x' + kx = F_0 \cos \Omega t.$$

This is a nonhomogeneous problem. That means we have to consider the homogeneous problem first. Fortunately, we've already done this in the previous lecture, so we get to use it here. As before, we'll break this up in different cases.

Remark: this same differential equation matters in a variety of different settings: mechanical systems such as springs, as discussed here; electrical systems with resistors, capacitors and solenoids, see below. In short, this differential equation is important to study in *any* setting where we encounter vibrations or oscillations.

1. NO DAMPING ($\gamma = 0$), NO RESONANCE ($\Omega \neq \omega_0$)

The differential equation is

$$mx'' + kx = F_0 \cos \Omega t$$

$$\Rightarrow \qquad x'' + \omega_0^2 x = \frac{F_0}{m} \cos \Omega t,$$

where $\omega_0^2 = k/m$ is the square of the natural frequency of the system.

(a) **The homogeneous solution** of this problem (see last lecture) is

$$x_H = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

and the frequency of these oscillations is ω_0.

These lecture notes are based on those of Dr. Bernard Deconinck at the University of Washington. They have been modified to fit this class.

(b) **The particular solution** can be found using the method of undetermined coefficients. We guess

$$x_p = A\cos\Omega t + B\sin\Omega t.$$

This guess looks like a good one, but we need to be careful: if $\Omega = \omega_0$, then the terms of the particular solution also appear in the homogeneous solution. In that case we need to multiply our guess by t and try again. We'll deal with this case separately later. So, for now: **assume that $\boldsymbol{\Omega} \neq \omega_0$**. In that case we substitute the above guess in the equation. With

$$x_p'' = -A\Omega^2\cos\Omega t - B\Omega^2\sin\Omega t,$$

we get

$$-A\Omega^2\cos\Omega t - B\Omega^2\sin\Omega t + \omega_0^2(A\cos\Omega t + B\sin\Omega t) = \frac{F_0}{m}\cos\Omega t$$

$$\Rightarrow \quad \begin{cases} -A\Omega^2 + \omega_0^2 A &= F_0/m \\ -B\Omega^2 + \omega_0^2 B &= 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} A &= \dfrac{F_0}{m(\omega_0^2 - \Omega^2)} \\ B &= 0. \end{cases}$$

We see immediately that there are problems with the solution if we were to allow $\Omega = \omega_0$. Good thing we excluded this! The particular solution is

$$x_p = \frac{F_0}{m(\omega_0^2 - \Omega^2)}\cos\Omega t.$$

(c) **The general solution** is given by

$$x = x_H + x_p = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{F_0}{m(\omega_0^2 - \Omega^2)}\cos\Omega t.$$

At this point, c_1 and c_2 may be determined from the initial conditions.

Let's impose some special initial conditions. These aren't really essential, but they make the calculations a bit easier. Let

$$x(0) = 0, x'(0) = 0.$$

You'll easily check that the corresponding solution is given by

$$x = \frac{F_0}{m(\omega_0^2 - \Omega^2)}(\cos\Omega t - \cos\omega_0 t).$$

(You checked this, right? Otherwise go back and do it NOW!) Using a trig identity (check this too!) this solution is rewritten as

$$x = \frac{2F_0}{m(\omega_0^2 - \Omega^2)}\sin\omega_1 t\sin\omega_2 t,$$

where

$$\omega_1 = \frac{\omega_0 - \Omega}{2}, \quad \omega_2 = \frac{\omega_0 + \Omega}{2}.$$

Assume that Ω is close to ω_0 (but not equal, otherwise the above result is not valid, remember?) then ω_1 is close to zero, which means that the factor $\sin\omega_1 t$ is a function with a frequency that is much smaller than that of $\sin\omega_2 t$. We can trivially rewrite our solution as

$$x = \left(\frac{2F_0}{m(\omega_0^2 - \Omega^2)} \sin \omega_1 t\right) \sin \omega_2 t = U(t) \sin \omega_2 t,$$

where

$$U(t) = \frac{2F_0}{m(\omega_0^2 - \Omega^2)} \sin \omega_1 t$$

is interpreted as a time-dependent amplitude: it is a function that is changing much slower than $\sin \omega_2 t$. This time-dependent amplitude is itself oscillating in time, but it takes a lot longer for it to come around. The kind of pattern we get is illustrated in Fig. 1.

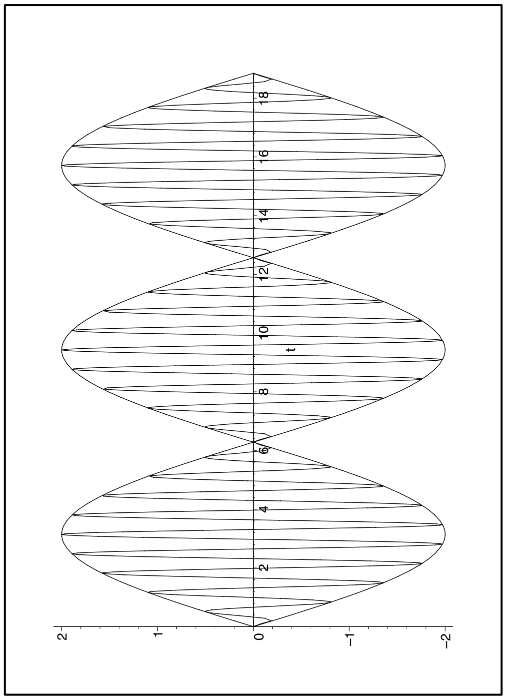

Figure 1: The beats phenomenon with $\Omega = 10$, $\omega_0 = 9$.

Such a signal is called a modulated wave, and the phenomenon observed is that of **beats**: there are two frequencies in this problem. The first frequency is the slow one, which governs the *modulation* of the amplitude. The second frequency is that of the underlying *carrier wave*, *i.e.*, the fast oscillations.

2. No damping ($\gamma = 0$), resonance ($\Omega = \omega_0$)

Let's look at one of the special cases we skipped. The solution given above is not valid when $\Omega = \omega_0$. What happens in this case?

The differential equation is

$$mx'' + kx = F_0 \cos \omega_0 t$$

$$\Rightarrow \qquad x'' + \omega_0^2 x = \frac{F_0}{m} \cos \omega_0 t.$$

(a) **The homogeneous solution** of this problem is the same as before:

$$x_H = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

(b) **The particular solution** can be found using the method of undetermined coefficients. We guess

$$x_p = A \cos \omega_0 t + B \sin \omega_0 t.$$

This guess is no longer valid, since both terms of our guess occur in the homogeneous solution. This implies we need to multiply our guess by t and try again. We get

$$\begin{aligned}
& x_p = At \cos \omega_0 t + Bt \sin \omega_0 t, \\
\Rightarrow \quad & x_p' = A \cos \omega_0 t + B \sin \omega_0 t - A\omega_0 t \sin \omega_0 t + B\omega_0 t \cos \omega_0 t, \\
\Rightarrow \quad & x_p'' = -2A\omega_0 \sin \omega_0 t + 2B\omega_0 \cos \omega_0 t - A\omega_0^2 t \cos \omega_0 t - B\omega_0^2 t \sin \omega_0 t.
\end{aligned}$$

Substituting this in the equation, we obtain

$$\begin{aligned}
& -2A\omega_0 \sin \omega_0 t + 2B\omega_0 \cos \omega_0 t - A\omega_0^2 t \cos \omega_0 t - B\omega_0^2 t \sin \omega_0 t + \\
& \qquad \omega_0^2 \left(At \cos \omega_0 t + Bt \sin \omega_0 t \right) = \frac{F_0}{m} \cos \omega_0 t \\
\Rightarrow \quad & -2A\omega_0 \sin \omega_0 t + 2B\omega_0 \cos \omega_0 t = \frac{F_0}{m} \cos \omega_0 t \\
\Rightarrow \quad & \left\{ \begin{array}{rcl} -2A\omega_0 & = & 0 \\ 2B\omega_0 & = & \dfrac{F_0}{m} \end{array} \right. \\
\Rightarrow \quad & \left\{ \begin{array}{rcl} A & = & 0 \\ B & = & \dfrac{F_0}{2m\omega_0}. \end{array} \right.
\end{aligned}$$

The particular solution is

$$x_p = \frac{F_0}{2m\omega_0} t \sin \omega_0 t.$$

(c) **The general solution** is given by

$$x = x_H + x_p = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t.$$

At this point, c_1 and c_2 may be determined from the initial conditions.

Let's think about this solution. After a significant time, the particular solution will give us the most important part, as it's linearly increasing in time, whereas the homogeneous solution is just oscillating from here to oblivion. So, what does this particular solution look like? It's plotted in Fig. 2. You observe that the amplitude of the solution is linearly growing in time. This phenomenon is called *resonance*.

It occurs whenever we force a system at its natural frequency. Resonance is one of the important elementary processes in all kinds of physical systems. You may imagine that this is not necessarily a good thing in applications: if we force the spring to oscillate at higher and higher amplitudes, it may eventually break! This gives us another way to think about the natural frequency of the system: it is the frequency that if we use it to force the system results in the system oscillating more and more wildly, eventually leading to breakdown, unless we have a way to prevent it. Preventing it is the subject of the next case.

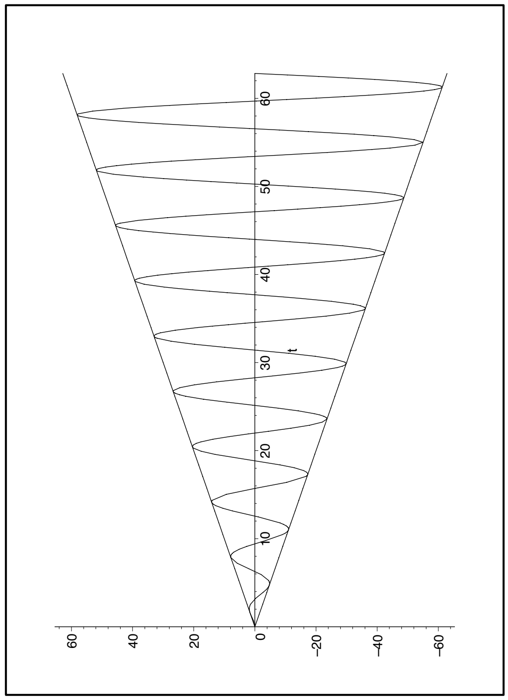

Figure 2: The phenomenon of resonance with $\omega_0 = 1$.

3. Damping ($\gamma \neq 0$)

Let's look at what happens when we include the effects of damping. In any realistic system some amount of damping will be present. Sometimes, its effects are so minuscule they may be ignored. In other cases, they may dominate.

The differential equation is

$$mx'' + \gamma x' + kx = F_0 \cos \omega_0 t.$$

(a) We've seen how to find the **homogeneous solution** of this problem in the previous lecture: assuming that we're dealing with subcritical damping we have

$$x_H = e^{-\gamma t/2m} \left(c_1 \cos \omega t + c_2 \sin \omega_0 t \right),$$

where $\omega = \sqrt{4km - \gamma^2}/2m$. Note that by assuming subcritical damping we've let $\gamma^2 < 4km$. As we've seen this corresponds to a damped oscillation. Thus, no matter what the particular solution is, or what the initial conditions are, we have

$$\lim_{t \to \infty} x_H = 0.$$

This implies that, if we wait sufficiently long, all the important information about the general solution is contained in the particular solution! So, what are we waiting for? Let's find it!

(b) **The particular solution** can be found using the method of undetermined coefficients, as before. We guess

$$x_p = A \cos \Omega t + B \sin \Omega t.$$

After substituting this guess in the equation and equating the coefficients of sine and cosine, and doing some algebra, we get (check this!):

$$\begin{cases} A &= \dfrac{\frac{F_0}{m}(\omega_0^2-\Omega^2)}{(\omega_0^2-\Omega^2)^2+\gamma_0^2\Omega^2}, \\ B &= \dfrac{\frac{F_0}{m}\gamma\Omega}{(\omega_0^2-\Omega^2)^2+\gamma_0^2\Omega^2}, \end{cases}$$

where $\gamma_0 = \gamma/m$. We see that the particular solution is always bounded as $t \to \infty$.

Even if we were to have $\Omega = \omega_0$, or $\Omega = \omega$, the particular solution we've constructed works just fine. Thus there's never a danger of the amplitude of the particular solution exploding on us, as there was in the resonant case without damping.

(c) **The general solution** is given by

$$x = x_H + x_p = e^{-\gamma t/2m}\left(c_1 \cos\omega t + c_2 \sin\omega_0 t\right) + A\cos\Omega t + B\sin\Omega t,$$

where A and B are given by the expressions above. At this point, c_1 and c_2 may be determined from the initial conditions.

Let's think about this solution. After a significant time, the particular solution will give us the only important part, as it's not decaying in time, whereas the homogeneous solution is. On the other hand, the particular solution is just an oscillation. What can we say about it? One of the most important aspects of an oscillation is its amplitude. For the particular solution here, that amplitude is given by (do I need to say it: Check it!)

$$\sqrt{A^2+B^2} = \frac{F_0/m}{\sqrt{\gamma_0^2\Omega^2+(\omega_0^2-\Omega^2)^2}}.$$

It is clear from this formula that the magnitude of the response of the system depends a lot on the parameters of the input forcing. To quantify that, we rewrite the above as

$$\frac{m}{F_0}\sqrt{A^2+B^2}\,\omega_0^2 = \frac{1}{\sqrt{\frac{\gamma_0^2}{\omega_0^2}\frac{\Omega^2}{\omega_0^2}+\left(1-\frac{\Omega^2}{\omega_0^2}\right)^2}}.$$

This expression is used to plot the amplitude response graph, shown in Fig. 3. This figure shows the scaled (by a factor m/F_0) amplitude of the response, as a result of forcing the system with frequency Ω (in units of ω_0), for different values of the normalized damping γ_0/ω_0. We see that for no damping, there is a vertical asymptote at $\Omega/\omega_0 = 1$, as expected. For non-zero damping, there is still a maximum in the amplitude near $\Omega/\omega_0 = 1$. Thus, if we want to get a lot from a little (and who doesn't?), we should force the system with a frequency that is close to its natural frequency, as this will maximize the amplitude of the output response.

Microwaves work on this principle: the microwave operates in the microwave regime (gee, coincidence?), which is close to the natural frequency for water molecules. Water is the main ingredient in any food. As a result of the microwave forcing, the water molecules vibrate a lot, giving off a lot of heat due to friction. It is this heat that warms your food.

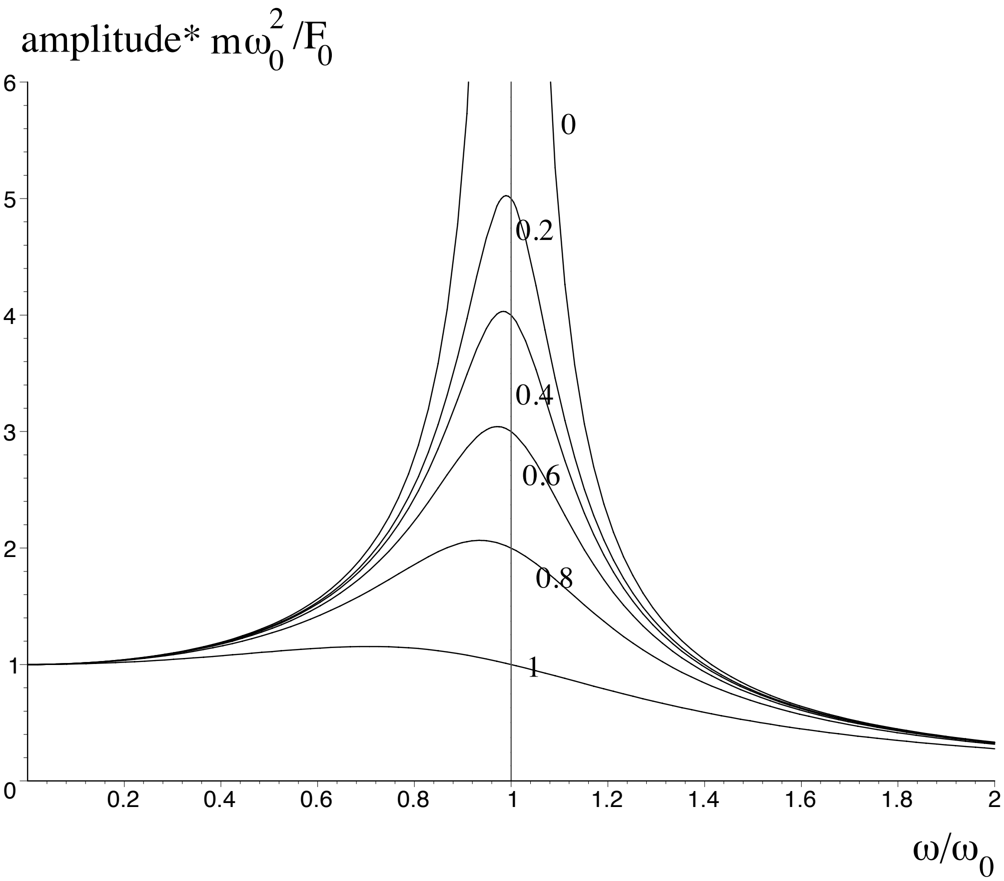

Figure 3: The amplitude-response graph for various values of γ_0/ω_0.