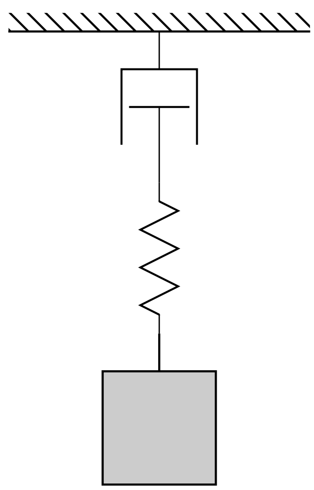
MATH 234 - MECHANICAL VIBRATIONS

PROBLEM SET-UP

As an application to second-order linear equations with constant coefficients, we go back to Newton's law:

$$F = ma.$$

Here F is the sum of the forces acting on the point particle of mass m, and a denotes the particle's acceleration. We'll consider the case of a particle suspended from a linear spring with spring constant k. The top of the spring could be moving in a prescribed way, and the particle is undergoing damping. You can think of damping as a consequence of dealing with a realistic spring (small damping) or a physical damper. The situation is described in the diagram below:



So, what's our governing equation? We need to determine the explicit form of the total force. We have

$$F = F_{\text{spring}} + F_{\text{damper}} + F_{\text{external}}.$$

What are the functional forms of these different forces. The last one is given to us as

$$F_{\text{external}} = F_e(t),$$

some function of t. The other two are not much harder. The damping force is

$$F_{\text{damper}} = -\gamma v,$$

where v is the velocity of the particle, and γ is a constant damping rate. Note that this force has a negative sign: it opposes the motion. The last force is given by Hooke's law:

$$F_{\text{spring}} = -kx.$$

This force also comes with a minus sign. It is a restoring force: it pulls the particle back to its equilibrium position.

Putting all these together, we finally obtain

$$\boxed{mx'' + \gamma x' + kx = F_e(t).}$$

Here we've used that $a = x''$, $v = x'$: the acceleration and the velocity are the second, respectively first, time derivative of the position.

Unforced oscillations - The Homogeneous Equation

If $F_e(t) \neq 0$ then the above differential equation is nonhomogeneous. As we've seen: whenever we're facing a nonhomogeneous problem, we should solve the homogeneous problem first. We'll get back to the nonhomogeneous problem when we talk about forced oscillations in the next lecture. Here we consider

$$mx'' + \gamma x' + kx = 0.$$

We refer to the motions predicted by this differential equation as *free* motions. Further, if $\gamma = 0$, the motion is undamped. Otherwise, if $\gamma > 0$, then the motion is damped.

We start by considering the characteristic equation:

$$m\lambda^2 + \gamma\lambda + k = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}.$$

Note that all of m, γ, and k are not allowed to be negative.

There are three possible cases:

1. $\gamma^2 > 4mk$: lots of damping. This is known as an *overdamped* spring.
2. $\gamma^2 = 4mk$: still a lot of damping, but less than overdamped. We call this a *critically damped* spring.
3. $\gamma^2 < 4mk$: a small amount of damping. This is known as the *underdamped* spring.

We'll spend most of our time studying the underdamped case. Note that the undamped spring is a special case of the underdamped spring.

Underdamped oscillations

If $\gamma^2 < 4mk$ then $4mk - c^2 > 0$, so that

$$\lambda_{1,2} = \frac{-\gamma \pm i\sqrt{4mk - \gamma^2}}{2m} = -\frac{\gamma}{2m} \pm i\omega,$$

where

$$\omega = \frac{4mk - \gamma^2}{2m}.$$

The general solution is given by

$$\begin{aligned} x &= c_1 e^{-\gamma t/2m} \cos \omega t + c_2 e^{-\gamma t/2m} \sin \omega t \\ &= e^{-\gamma t/2m}(c_1 \cos \omega t + c_2 \sin \omega t). \end{aligned}$$

Let's look at this solution in two different cases.

1. **The undamped spring: $\gamma = 0$.** In this case the exponential dissapears and

   $$x = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

   with

   $$\omega_0 = \frac{\sqrt{4mk}}{2m} = \frac{\sqrt{mk}}{m} = \sqrt{\frac{k}{m}}.$$

   The parameter ω_0 is called the *natural frequency* of the system: it is the frequency the spring-particle system likes to oscillate at when no other forces (external, damping) are present. In order to completely determine the solution, we need initial conditions to specify the constants c_1 and c_2. Often it is useful to rewrite the solution formula in so-called *amplitude-phase* form. Let

   $$\begin{cases} c_1 &= A \cos \varphi \\ c_2 &= A \sin \varphi \end{cases}.$$

   Then

   $$A = \sqrt{c_1^2 + c_2^2}, \quad \tan \varphi = \frac{c_2}{c_1}.$$

   We have

   $$\begin{aligned} x &= A \cos \varphi \cos \omega_0 t + A \sin \varphi \sin \omega_0 t \\ &= A \cos(\omega_0 t - \varphi). \end{aligned}$$

   The new parameters A and φ are called the *amplitude* and the *phase* respectively, of the solution. We see that the solution is periodic with period

   $$T = \frac{2\pi}{\omega_0}.$$

   A plot of an undamped solution is shown in Fig. 1.

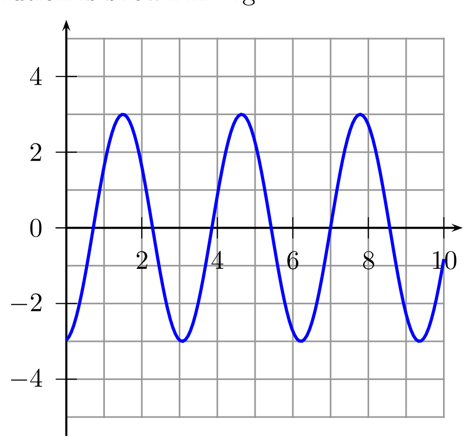


Figure 1: A solution of an undamped system: $x = 3\cos(2t - 3)$.

2. **The underdamped spring: $\gamma > 0$.**

   In the underdamped case with $\gamma > 0$ we have

$$\begin{aligned} x &= e^{-\gamma t/2m}(c_1 \cos \omega t + c_2 \sin \omega t) \\ &= Ae^{-\gamma t/2m} \cos(\omega t - \varphi). \end{aligned}$$

   We see that the factor $Ae^{-\gamma t/2m}$ plays the role of a time-dependent amplitude. If the damping rate γ is small, then this amplitude factor will decay to zero, but at a slow rate.

While these solutions are not periodic in the traditional sense, we can still define a *quasi-period* and *quasi-frequency* that will give us an indication of how often the amplitude flips from positive to negative values.

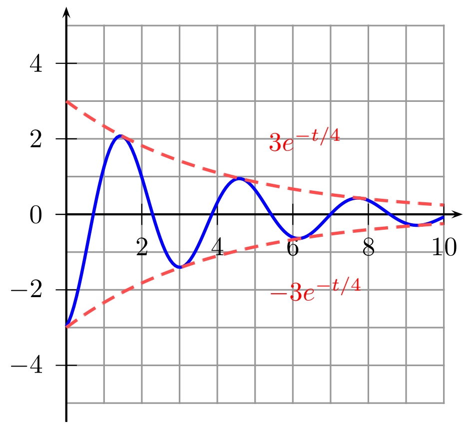


Figure 2: A solution of an underdamped system: $x = 3e^{-t/4}\cos(2t - 3)$. The red dashed line represents the "solution" envelope given by the curves $3e^{-t/4}$ and $-3e^{-t/4}$.

If we compare the ratio of $\frac{\omega}{\omega_0}$ we see that for small γ, we can expand the expression as a Taylor-series

$$\frac{\omega}{\omega_0} = \frac{\sqrt{4mk - \gamma^2}/2m}{\sqrt{k/m}} = \sqrt{1 - \frac{\gamma^2}{4mk}} \approx 1 - \frac{1}{8mk}$$

If we look at ω, we find that the quasi-frequency is given by

$$\omega \approx (1 - \frac{1}{8mk})\omega_0$$

so that the frequency <u>with</u> damping is lower than that without damping. Likewise, the quasi-period is given by

$$T_d \approx (1 + \frac{1}{8mk})T$$