$$\boxed{\begin{array}{c} \text{MATH 234 - ABEL'S THEOREM} \\ \text{AND REDUCTION OF ORDER} \end{array}}$$

RECAP

In the last lecture, we discussed that the general solution of the ode $ay'' + by' + cy = 0$ is given by $y = c_1 y_1 + c_2 y_2$ if y_1 and y_2 were linearly independent solutions to the ode. We will refer to the linearly independent solutions y_1 and y_2 as **fundamental solutions**.

We also argued that we will need two linearly independent solutions in order to solve a second-order initial value problem since we know we will have two constants of integration from removing the second order derivative.

In addition, we showed that two functions $f(x)$ and $g(x)$ are linearly independent if their **Wronskian** is not zero for all x where the Wronskian is defined as

$$W(f,g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - f'(x)g(x) \tag{1}$$

Combining this information, we can rephrase our superposition theorem from last time as:

THEOREM

If y_1 and y_2 are two solutions of the differential equation

$$ay'' + by' + cy = 0$$

and their Wronskian

$$W(y_1, y_2) \neq 0,$$

then the general solution to the differential equation is

$$y = c_1 y_1 + c_2 y_2.$$

ABEL'S THEOREM AND REDUCTION OF ORDER

Consider the following example:

Find the fundamental solution set to the differential equation

$$y'' - 2y' + y = 0, \qquad y(0) = 1, \ \ y'(0) = 2$$

To find the fundamental solution set, we need to find two linearly independent functions that are solutions to the above differential equation. Since this is a constant coefficient problem, we can guess that the solution is of the form $y = e^{\lambda x}$. However, if we plug this into the ode to find the characteristic equation, we have

$$\lambda^2 - 2\lambda + 1 = 0 \qquad \Rightarrow \qquad \lambda = 1$$

Thus, we only get one solution e^x. We need to have one additional fundamental solution to the ode that is linearly independent to e^x. Therefore, we do not know how to solve this differential equation... yet!

We have already seen how to solve constant-coefficient equations when the roots of the characteristic equation are real and different.

In this lecture, well see how to solve the case where the two roots λ_1 and λ_2 are equal: $\lambda_1 = \lambda_2$ (note that they are automatically real in this case). It is clear that our previous plan of attach will not work, since we now have $y_1 = y_2$, and we do not have two linearly independent solutions.

Let y_1 and y_2 be any two solutions of

$$y'' + p(x)y' + q(x)y = 0,$$

then

$$W(y1, y2) = ce^{-\int p(x)\,dx},$$

where c is a constant.

By definition of the Wronskian, $W(y_1, y_2) = y_1 y_2' - y_1' y_2$. We can also take the derivative of the Wronskian to find

$$W'(y_1, y_2) = y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2'$$

Using this definition, along with the fact that both y_1 and y_2 satisfy the ODE $y'' + p(x)y' + q(x)y = 0$, we have

$$
\begin{aligned}
W'(y_1, y_2) &= y_1 y_2'' - y_1'' y_2' \\
&= y_1 \left(-(p(x)y_2' + q(x)y_2) \right) - y_2 \left(-(p(x)y_1' + q(x)y_1) \right) \\
&= -p(x)\left(y_1 y_2' - y_2 y_1' \right) + q(x)\left(y_1 y_2 - y_1 y_2 \right) \\
&= -p(x)W(y_1, y_2)
\end{aligned}
$$

Thus, we have an ODE for the Wronskian which we can solve by separation.

$$
W' = -p(x)W \quad \implies \quad W(y_1, y_2) = ce^{-\int p(x)\ dx}
$$

where c is a constant. Thus, we have proved our final statement. ★

So, here's the million dollar questions. How can we actually use this theorem to solve our problem of knowing only one fundamental solution?

REDUCTION OF ORDER

We can now use Abels theorem to get a second linearly independent solution of a second-order linear differential equation if we already know a first one. This is known as *reduction of order*, because it reduces the problem of finding a solution of a second-order equation to that of solving a related first-order equation. Heres how this works: supposed we know y_1, but we do not know y_2.

With Abel's theorem in mind, we have two ways to write an expression for the Wronskian of the fundamental solutions. One from the definition (given by Equation (1)) and the other from Abel's theorem. Thus, we have the relationship:

$$
W(y_1, y_2) = y_1 y_2' - y_1' y_2 = ce^{-\int p(x)\ dx} \tag{2}
$$

If we want to find the second linearly independent solution, we will use what we know about the differential equation ($p(x)$) to find the Wronskian, and use the first solution y_1 to find the second linearly independent solution.

Let's try this now with out initial example problem: $y'' - 2y' + y = 0$.

EXAMPLE

If $y(x) = e^x$ is a fundamental solution to the differential equation

$$y'' - 2y' + y = 0,$$

find the second fundamental solution.

SOLUTION

Since the ODE is already in standard form, we know that the Wronskian between the two fundamental solutions is given by

$$W = ce^{-\int -2dx} = ce^{2x}.$$

We also know from the definition of the Wronskian that

$$W = y_1 y_2' - y_1' y_2 \quad \Rightarrow \quad W = e^x y_2' - e^x y_2,$$

where we have used the fact that $y_1 = e^x$ and $y_1' = e^x$. Equating the two expressions for the Wronskian, we have

$$e^x y_2' - e^x y_2 = ce^{2x} \quad \Rightarrow \quad y_2' - y_2 = ce^x.$$

Now we have a **first-order** linear differential equation for the second fundamental solution. Hence, we have reduced the order of the differential equation (reduction of order).

To solve $y_2' - y_2 = ce^x$, we use a linear integrating factor which is given by $\mu(x) = e^{-x}$. This allows us to write the differential equation as

$$\frac{d}{dx}\left(e^{-x}y_2\right) = ce^x e^{-x} = c$$

Integrating the above and solving for y_2, we find

$$y_2(x) = cxe^x + \tilde{c}e^x,$$

where $\tilde{c}$ is another arbitrary constant. Since $\tilde{c}$ is multiplying our first fundamental solution **only**, we can ignore this term (it's a constant multiple of a known fundamental solution). Thus, the second fundamental solution is given by

$$\boxed{y_2(x) = cxe^x}$$

Guess what? Abel's theorem and reduction of order is not just limited to constant coefficient problems. Consider the following example:

Consider the equation
$$2x^2y'' + 3xy' - y = 0,$$
for $x > 0$.

(a) Check that $y_1 = 1/x$ is a solution of this equation

(b) Find a second fundamental solution to the ODE.

SOLUTION

(a) First, let's check that $y = 1/x$ is a solution to the ODE. To do that, we need to calculate y' and y'':
$$y' = \frac{-1}{x^2}, \qquad y'' = \frac{2}{x^3}$$

Plugging this into the ODE, we have
$$
\begin{aligned}
2x^2y'' + 3xy' - y &= 2x^2\frac{2}{x^3} + 3x\frac{-1}{x^2} - \frac{1}{x} \\
&= \frac{4}{x} - \frac{3}{x} - \frac{1}{x} \\
&= 0 \quad \bigstar
\end{aligned}
$$

Thus $y_1 = 1/x$ is a solution.

(b) Now we use the method above to find a second solution y_2, linearly independent of the first one. First we need to compute the Wronskian W. Using Abels theorem, we have that
$$W = ce^{\int -p(x)\ dx}$$

For our equation $p(x) = 3x/(2x^2) = 3/(2x)$, since we need to write the differential equation so that the coefficient of y'' is one, in order to use Abels theorem. Thus
$$W = ce^{\int -3/(2x)\ dx} = ce^{-3/2\ln(x)} = cx^{-3/2}$$

There's no need to choose c at this point. Since $y_1(x) = x^{-1}$, $y_1' = -x^{-2}$. Combining these facts with Abel's theorem and the definition of the Wronskian, we find
$$x^{-1}y_2' + x^{-2}y_2 = cx^{-3/2} \quad \Rightarrow \quad y_2' + x^{-1}y_2 = cx^{-1/2}.$$

Again, we now have a first-order linear differential equation for y_2. The integrating factor is given by $\mu(x) = x$ and thus, we find
$$\frac{d}{dx}(xy_2) = cx^{1/2}.$$

Integrating and solving for y_2, we find
$$\boxed{y_2(x) = 2cx^{1/2} = 2c\sqrt{x}}$$

Now, let's consider another example: it will allow us to solve the case of linear, constant-coefficient equations where the roots of the characteristic equation are equal.

Consider the differential equation

$$ay'' + by' + c = 0 \implies a\lambda^2 + b\lambda + c = 0$$

Solving for λ we have

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a}$$

Let's assume that we have repeated roots. In other words, we only obtain one real, unique solution for λ. Find the two fundamental solutions, and then the general solution

SOLUTION

From algebra, we know that if we have repeated roots, the discriminant is identically zero. In other words, we know $b^2 - 4ac = 0$.

Since we only get one root, we only have one solution which is $y_1 = e^{-(b/(2a))x} = e^{\lambda x}$. **NOTE:** *For simplicity, let $\lambda = -b/(2a)$. Thus, $b/a = 2\lambda$.*

However, since this is a second-order ode, we expect that we will need two linearly independent solutions to obtain our general solution. To find the second solution, we will use reduction of order.

First, we use Abel's theorem to calculate the Wronskian. Note that we have $p(x) = b/a = 2\lambda$. Also, let's use the constant α in Abel's theorem, as the equation already has a c in it. Abel's theorem gives us the Wronskian by the formula:

$$W = \alpha e^{-\int 2\lambda \ dx} = \alpha e^{-2\lambda x}$$

Since $y_1(x) = e^{\lambda x}$, $y_1' = \lambda e^{\lambda x}$. Combining these facts with Abel's theorem and the definition of the Wronskian, we find

$$e^{\lambda x} y_2' - \lambda e^{\lambda x} y_2 = \alpha e^{-2\lambda x}.$$

Writing in standard linear form for first-order differential equations, we have

$$y_2' + \lambda y_2 = \alpha e^{-\lambda x}.$$

Again, we now have a first-order linear differential equation for y_2. The integrating factor is given by $\mu(x) = e^{\lambda x}$ and thus, we find

$$\frac{d}{dx}\left(e^{\lambda x} y_2\right) = \alpha.$$

Integrating and solving for y_2, we find

$$\boxed{y_2(x) = \alpha x e^{\lambda x},}$$

and thus, the general solution is given by

$$\boxed{y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}.}$$

EXAMPLE

Consider the differential equation $y'' + 8y' + 16y = 0$ Find the general solution to the above ODE.

SOLUTION

Since we're solving a constant coefficient ODE, we can immediately jump to the characteristic equation:

$$\lambda^2 + 8\lambda + 16 = 0$$

and thus $\lambda_1 = \lambda_2 = -4$. Since the roots are repeated, we only get one fundamental solution.

However, using the above example, we can write down BOTH fundamental solutions! Thus $y_1 = e^{-4x}$, and $y_2 2 = xe^{-4x}$, using the result from the previous example. The general solution is

$$y = c_1 y_1 + c_2 y_2 = (c_1 + c_2 x)e^{-4x}$$

REDUCTION OF ORDER (ALTERNATIVE APPROACH)

There is another method called "Reduction of Order" due to D'Alambert. This basic idea is the same:

> *If we know one of the fundamental solutions, we can find the second fundamental solution by reducing the second order ODE to a first order ODE.*

However, the implementation of the alternative approach is more straight-forward. D'Alambert proposed that we consider the y_2 to be related to y_1 by

$$y_2 = v(x)y_1$$

In other words, the second solution is some unknown function $v(x)$ times the first known solution.

Consider the following example:

Find the second fundamental solution to the differential equation

$$(x-1)y'' - xy' + y = 0, \quad x > 0,$$

if the first fundamental solution is given by $y_1 = e^x$.

SOLUTION

Since $y_1(x) = e^x$ is a solution, let's find the second solution by plugging in $y_2(x) = v(x)e^x$. This means

$$y_2' = v'e^x + ve^x, \quad \text{and} \quad y_2'' = v''e^x + 2v'e^x + ve^x.$$

Plugging this expression into the original ODE, we get

$$
\begin{aligned}
(x-1)y'' - xy' + y &= (x-1)\left(v''e^x + 2v'e^x + ve^x\right) - x\left(v'e^x + ve^x\right) + ve^x \\
\\
&= v''\left((x-1)e^x\right) + v'\left(2(x-1)e^x - xe^x\right) + v\left((x-1)e^x - xe^x + e^x\right) \\
&= v''\left((x-1)e^x\right) + v'\left((x-2)e^x\right)
\end{aligned}
$$

Thus, if we assume the form $y_2(x) = v(x)y_1(x)$, then $v(x)$ must satisfy the differential equation

$$v''(x-1) + v'\left(2(x-1)e^x - xe^x\right) = 0.$$

Now, if we let $u(x) = v'(x)$, we can turn this into a first order ode for $u(x)$. In other words,

$$u'\left((x-1)e^x\right) + u\left(x-2\right)e^x = 0$$

. If we can solve for $u(x)$, we can find $v(x)$ by taking the integral and be done with it all.

$$u' = -\frac{(x-2)u}{x-1} \quad \Rightarrow \quad \ln(u(x)) = \ln(x-1) - x + \tilde{c}.$$

It helps to rewrite the integrand using the following trick:

$$-\frac{x-2}{x-1} = -\frac{x-1-1}{x-1} = -\left(\frac{x-1}{x-1} - \frac{1}{x-1}\right) = -1 + \frac{1}{x-1}.$$

Thus, $u(x) = c(x-1)e^{-x}$ and

$$v(x) = \int u(x)\, dx = -cxe^{-x} + d.$$

Thus, the second fundamental solution $y_2(x) = v(x)y_1(x)$ is given by

$$y_2(x) = \left(-cxe^{-x} + d\right)e^x = -cx + de^x$$

$$\boxed{y_2(x) = -cx + de^x}$$

Note: we could have just given $y_2(x) = x$. Also, you would get the same answer if you used Abel's theorem to help you out!