MATH 234 -THE WRONSKIAN
AND LINEAR INDEPENDENCE

RECAP

In the previous lecture, we learned how to solve

$$ay'' + by' + cy = 0,$$

using three steps:

1. Guess the form of the solution $e^{\lambda x}$ to determine the *characteristic equation*: $a\lambda^2 + b\lambda + c = 0$
2. Solve the characteristic equation to find the *fundamental solutions*: $y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}$
3. Combine the fundamental solutions (via superposition theorem) to determine the *general solution*: $y = c_1 y_1 + c_2 y_2$

It turns out that we can use this process to solve any linear constant-coefficient ODE of any order! However, so far, we have restricted ourselves to constant-coefficient 2nd order ODE whose characteristic equation does not have repeated roots.

If we slightly generalize the class of ODEs that we are considering to *any* linear second order ODE of the form

$$y'' + p(x)y' + q(x)y = 0,$$

the superposition theorem will still hold. Thus, the general solution is given by $y = c_1 y_1 + c_2 y_2$, where y_1 and y_2 are two linearly independent solutions of

$$y'' + p(x)y' + q(x)y = 0.$$

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But before we talk about linear independence of fundamental solutions, perhaps a more basic questions we should ask is the following:

> *If $y'' + p(x)y' + q(x)y = 0$, and we're given initial conditions $y(x_0) = y_0$ and $y'(x_0) = y'_0$, how do we even know if a solution will exist?*

In this lecture, we will talk about the existence of solutions, the concept of linear independence, and the Wronskian.

EXISTENCE, UNIQUENESS, AND THE WRONSKIAN

THEOREM:

Consider the initial value problem

$$y'' + p(x)y' + q(x)y = g(x), \qquad y(x_0) = y_0, y'(x_0) = y_0',$$

where $p(x)$, $q(x)$ and $g(x)$ are continuous on an open interval I that also contains x_0[1]. Then, there exists exactly one solution $y = \phi(x)$ of this problem and the solution exists throughout the interval I.

It's important to note the three things that this theorem says:

1. The initial value problem has *a* solution; in other words, a solution *exists*.
2. The initial value problem has *only one* solution; in other words, the solution is *unique*
3. The solution $\phi(x)$ is defined *throughout* the interval I where the coefficients are continuous and is at least twice differentiable there.

EXAMPLE:

Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)\frac{d^2y}{dt^2} + t\frac{dy}{dt} - (t+3)y = 0, \qquad y(1) = 2, \qquad \left.\frac{dy}{dt}\right|_{t=1} = 1$$

In this problem, if we write it in the form where the coefficient of the second derivative term is one, we find that $p(t) = 1/(t-3)$, $q(t) = -(t+3)/(t^2-3t)$, and $g(t) = 0$. The only points of discontinuity of the coefficients are at $t = 0$ and $t = 3$. Therefore, the longest open interval containing the initial point $t = 1$ in which all of the coefficients are continuous is $I = 0 < t < 3$. Thus, this is the longest interval in which our theorem guarantees that a solution exists.

LINEAR INDEPENDENCE AND THE WRONSKIAN

Now, let's return to a question we posed earlier:

> *Assume that both y_1 and y_2 are two solutions of*
>
> $$y'' + p(x)y' + q(x)y = 0.$$
>
> *Superpositions tells me that $y = c_1y_1 + c_2y_2$ is the* general solution *of the ODE if y_1 and y_2 are linearly independent. But how can we determine if two solutions are linearly independent?*

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[1] I is just a range of x values where the functions p, q, and g are well behaved.

Definition:

Two functions $f(x)$ and $g(x)$ are called *linearly independent* if the equation

$$c_1 f(x) + c_2 g(x) = 0, \quad \text{for all } x,$$

can only be satisfied by choosing $c_1 = 0$ and $c_2 = 0$. Two functions that are not linearly independent are called *linearly dependent.*

This is very similar to the concept for linearly independent vectors. For example, consider the two vectors v_1 and v_2. These vectors are linearly independent if $c_1 v_1 + c_2 v_2 = 0$ implies that both c_1 and c_2 are zero. If you need more verification, consider the concrete example where $v_1 = [10]^T$ and $v_2 == [01]^T$. It should be clear that they are linearly independent vectors. Furthermore, the only way that we can get the zero vector if we take a linear combination is by letting both c_1 and c_2 be identically zero.

Example:

$f(x) = e^x$ and $g(x) = 2e^x$ are linearly dependent because

$$-2f(x) + g(x) = 0,$$

so $c_1 = -2$ and $c_2 = 1$. If the only choice was to choose them both zero, the functions would be independent.

Wouldn't it be nice if there was an easier way to determine linear independence? Well, there is! We need to introduce the Wronskian first.

Definition:

The *Wronskian* of two function $f(x)$ and $g(x)$ is just the quantity

$$W(f,g)(x) = f(x)g'(x) - f'(x)g(x)$$

Example:

Let $f(x) = \sin(x)$ and $g(x) = \cos(x)$, find $W(f,g)(x)$.

$$W(f,g)(x) = \sin(x)(-\sin(x)) - \cos(x)(\cos(x)) = -1$$

It's sometimes easier to think of the Wronskian using matrix notations. In other words:

$$W(f,g)(x) = \det \begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} = f(x)g'(x) - f'(x)g(x) \quad \bigstar$$

Theorem

Two functions $f(x)$ and $g(x)$ are linearly dependent if their Wronskian

$$W(f,g)(x) = f(x)g'(x) - f'(x)g(x) = 0.$$

Proof: If $f(x)$ and $g(x)$ are linearly dependent, then we can find constants c_1 and c_2, not both zero, so that

$$c_1 f(x) + c_2 g(x) = 0, \text{ for all } x.$$

Then also

$$c_1 f'(x) + c_2 g'(x) = 0, \text{ for all } x.$$

Now, let's assume that $f \neq 0$, otherwise we can switch the roles of f and g. Then we can solve to find

$$c_1 = -\frac{c_2 g(x)}{f(x)}$$

$$\begin{aligned} c_1 f'(x) + c_2 g'(x) &= -\frac{c_2 g(x)}{f(x)} f'(x) + c_2 g'(x) \\ &= \frac{c_2}{f(x)} \left(g(x) f'(x) - g'(x) f(x) \right) = 0 \end{aligned}$$

Note that $c_2 \neq 0$, since otherwise c_1 would also be zero, which would imply the functions are linearly independent. Thus plugging the expression into the equation for the derivative, we have

$$fg' - f'g = 0 \Longrightarrow W(f, g)(x) = 0,$$

which is what we had to prove.

EXAMPLE:

Calculate the Wronskian for the functions $f(x) = e^x$ and $g(x) = 2e^x$ to determine if they are linearly independent.

$$W(e^x, 2e^x) = e^x(2e^x) - e^x(2e^x) = 0,$$

Since the Wronskian is equal to zero, the two functions are linearly dependent.

EXAMPLE:

Calculate the Wronskian for the functions $f(x) = e^x$ and $g(x) = e^{-x}$ to determine if they are linearly independent.

$$W(e^x, e^{-x}) = e^x(e^{-x}) - e^x(-1e^{-x}) = 1 - (-1) = 2,$$

Since the Wronskian is not equal to zero, the two functions are linearly independent.