MATH 234 - 2ND ORDER ODES
CONSTANT COEFFICIENTS

In this lecture, we'll look at second-order equations for the first time. All the second-order equations we'll consider here will be linear. We won't look at nonlinear equations again until we get to nonlinear systems, much later in these notes.

Any second-order linear equation is of the form

$$r(x)y'' + p(x)y' + q(x)y = g(x),$$

where $r(x)$, $p(x)$ and $q(x)$ may be functions of x. For now, we'll assume they are constants: $r(x) = a$, $p(x) = b$ and $q(x) = c$, with a, b and c constant. Further, we'll start with the **homogeneous case**, *i.e.*, the case where $g(x) = 0$. Thus, we'll consider

$$ay'' + by' + cy = 0,$$

where $y = y(x)$ is the function we're looking for. Let's introduce some shorthand. Let

$$L[y] = ay'' + by' + cy,$$

so that the differential equation simply is $L[y] = 0$. Let's discuss a few properties of this equation.

THEOREM: PRINCIPLE OF SUPERPOSITION

If y_1 and y_2 are independent solutions of this equation, then $y(x) = c_1y_1(x) + c_2y_2(x)$ is the general solution.

Proof: Note that the general solution will depend on two constants, since we are now dealing with a second-order equation.

$$\begin{aligned} L[y] &= ay'' + by' + cy \\ &= a(c_1y_1'' + c_2y_2'') + b(c_1y_1' + c_2y_2') + (c_1y_1 + c_2y_2) \\ &= c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2) \\ &= c_1L[y_1] + c_2L[y_2] \\ &= c_10 + c_20 \\ &= 0. \end{aligned}$$

We've used that $L[y_1] = 0$ and $L[y_2] = 0$, since y_1 and y_2 are solutions. Thus $y_1 = c_1y_1 + c_2y_2$ is also a solution, which is what we had to prove. ★

We can use this theorem to get new solutions from known ones: if y_1 and y_2 are solutions, then so are $y_3 = (y_1 + y_2)/2$ and $y_2 = (y_1 - y_2)/2$. These are easily obtained by choosing $c_1 = c_2 = 1/2$, and $c_1 = 1/2$, $c_2 = -1/2$ in the theorem.

In order for the theorem to hold, y_1 and y_2 have to be "independent". What does this mean? We'll define this properly soon, but for the moment it suffices to say that y_1 and y_2 are not a multiple of each other. If this happens, say $y_2 = \alpha y_1$, for some constant α, then

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 y_1 + c_2 \alpha y_1 \\
&= (c_1 + c_2 \alpha) y_1 \\
&= c_3 y_1,
\end{aligned}
$$

where $c_3 = c_1 + c_2\alpha$ is another constant. We see that in this case, our proposed general solution y only depends on one constant. That's not enough!

Here's why this theorem absolutely rocks: in order to find the general solution of

$$L[y] = 0,$$

it suffices to find two solutions y_1 and y_2! Awesome!

It's easy to find two such solutions: guess

$$y = e^{\lambda x},$$

for some constant λ, to be determined. Then

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}.$$

Plugging all this in, we get

$$
\begin{aligned}
a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + c e^{\lambda x} &= 0 \\
e^{\lambda x}\left(a\lambda^2 + b\lambda + c\right) &= 0 \\
a\lambda^2 + b\lambda + c &= 0,
\end{aligned}
$$

since $e^{\lambda x}$ is never zero. Thus, in order to find solutions, we have to choose λ to be a solution of the quadratic equation

$$\boxed{a\lambda^2 + b\lambda + c = 0.}$$

This equation is known as the **Characteristic equation** of the differential equation. From it, we get two solutions for λ:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This gives two solutions of the original differential equation, namely

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}.$$

Using our theorem, we find that the general solution is

$$\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.}$$

Thus, we've constructed the general solution for a second-order linear equations with constant coefficients, and all we've had to do was solve a quadratic equation!

This works very well if λ_1 and λ_2 are both real, and different. In the other cases, we'll have to do a bit of extra work. Let's look at some examples where the above does work.

Example

Find the solution to the IVP:

$$y'' - y = 0, \qquad y(0) = 2, \quad y'(0) = -1,$$

Solution

Note that we're specifying two initial conditions, since we have two constants to determine. Let's start with the characteristic equation: we have $a = 1$, $b = 0$, $c = -1$.

$$\lambda^2 - 1 = 0 \quad \Rightarrow \lambda_1 = 1, \;\; \lambda_2 = -1,$$

from which $y_1 = e^x$, $y_2 = e^{-x}$, and the general solution is

$$y = c_1 e^x + c_2 e^{-x}.$$

Since we'll need y' to use the second initial condition, let's calculate it now: $y' = c_1 e^x - c_2 e^{-x}$. Plugging in the two initial conditions, we get

$$y(0) = c_1 + c_2, \quad y'(0) = c_1 - c_2,$$

so that $c_1 + c_2 = 2$ and $c_1 - c_2 = -1$. Adding and subtracting these two equations we find that $c_1 = 1/2$ and $c_2 = 3/2$. Finally, the solution of the initial-value problem is

$$y = \frac{1}{2}e^x + \frac{3}{2}e^{-x}.$$

Example

Find the general solution to the ode:

$$y'' + 5y' + 6y = 0.$$

Solution

The characteristic equation is

$$\begin{aligned} & \lambda^2 + 5\lambda + 6 = 0 \\ \Rightarrow \quad & (\lambda + 2)(\lambda + 3) = 0 \\ \Rightarrow \quad & \lambda_1 = -2, \quad \lambda_2 = -3, \end{aligned}$$

and thus $y_1 = e^{-2x}$, $y_2 = e^{-3x}$. The general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-2x} + c_2 e^{-3x}.$$