$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \\ a_{n1} & & a_{nn} \end{bmatrix} \qquad \underset{\text{row \#}}{a}{}_{rc} \leftarrow \text{column \#}$$

Please Review

① How to calculate the determinant of a general matrix

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}M_{11} + a_{12}M_{12} + a_{13}M_{13}$$

Where M_{ij} is the determinant of the matrix when the i-th row and j-th column is eliminated

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Alt: $= a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}$

In general, for an $n \times n$ matrix

$$\det(A) = a_{11} \cdot M_{11} - a_{12}M_{12} + a_{13}M_{13} - a_{14}M_{14} + \cdots$$

Notice ... alternating signs.

Short cuts: If A is a triangular matrix, then $\det(A) =$ product of diagonal.

$$I = \text{identity} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & & \\ 0 & & \ddots & \end{bmatrix}$$

$A\vec{x} = \vec{b}$ has unique sol if $\det(A) \neq 0$

$A\vec{x} = \vec{0}$ has non-zero / non-trivial sol if $\det(A) = 0$

② taking the inverse of a Matrix.

$A \cdot A^{-1} = I$

For 2×2, it's easy. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^{-1} = \frac{1}{\det A} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In general

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} M_{11} & -M_{12} & M_{13} & \cdots \\ -M_{21} & M_{22} & -M_{23} & \cdots \\ \vdots & \vdots & & \end{bmatrix}^T$$

$1/\det(A) \cdot$ Matrix of cofactors transposed!

An $n\times n$ matrix A has an inverse if and only if $\det(A) \neq 0$

Systems of Differential Equations

Any n-th order diff. eq. can be written as a system of first order ODEs