Example Find the inverse transform of

$$H(s)=\frac{4}{s^4+4s^2}$$

$$H(s)=\frac{4}{s^2(s^2+4)}=\underbrace{\frac{2}{s^2}}_{F(s)}\cdot\underbrace{\frac{2}{s^2+4}}_{G(s)}$$ → makes life easy!

→ $f(t)=\mathcal{L}^{-1}\{F(s)\}=2t$ → via #3

→ $g(t)=\mathcal{L}^{-1}\{G(s)\}=\sin(2t)$

→ via the thm:

$$h(t)=f*g=\int_0^t 2(t-\tau)\cdot\sin(2\tau)\,d\tau$$

$$=\int_0^t\left[2t\sin(2\tau)-2\tau\sin(2\tau)\right]d\tau$$

$$=-t\cos(2\tau)-\tfrac{1}{2}\sin(2\tau)+\tau\cos(2\tau)\Big|_0^t$$

$$=-t\cos(2t)-\tfrac{1}{2}\sin(2t)+t\cos(2t)$$
$$-[-t\cdot 1-0+0]$$

→ $$=\boxed{t-\tfrac{1}{2}\sin(2t)}$$

Why useful? → you can derive "variation of parameters" using this method

Solve: $y''+y=g(t)$ for a general $g(t)$! $y(0)=y_0$ $y'(0)=y_0'$

Could use variation of parameters, but the proof is much easier w/ Laplace!

$$s^2Y-sy_0-y_0'+Y=G(s)$$

$$(s^2+1)Y=G(s)+sy_0+y_0'$$

$$Y=G(s)\cdot\frac{1}{s^2+1}+\frac{sy_0}{s^2+1}+\frac{y_0'}{s^2+1}$$

$$y(t)=y_0'\sin(t)+y_0\cos(t)+\int_0^t\sin(t-\tau)g(\tau)d\tau$$

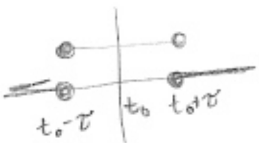
Special properties of the dirac function:

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = f(t_0) \qquad \int_{0}^{\infty} \delta(t-t_0) f(t)\,dt = f(t_0) \text{ if } t_0 \geq 0$$

proof:

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = \lim_{\tau \to 0} \frac{1}{2\tau} \cdot \int_{t_0-\tau}^{t_0+\tau} f(t)\,dt$$

→ mean value theorem



$$= \lim_{\tau \to 0} \frac{1}{2\tau} \cdot 2\tau \cdot f(t^*) \quad \rightarrow t^* \in (t_0-\tau, t_0+\tau)$$

$$= f(t^*) \quad \text{but as } \tau \to 0, \ t^* \in [t_0, t_0)$$

$$\boxed{t^* \to t_0}$$

LaPlace transform $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$

proof: $\mathcal{L}\{\delta(t-t_0)\} = \int_0^{\infty} e^{-st} \cdot \delta(t-t_0)\,dt$

$= e^{-st_0}$ ✓ by above!

Ex Solve $2y'' + y' + 2y = \delta(t-5)$ $\quad y(0) = y'(0) = 0$

Step 1 Take Laplace transform

$$2s^2 Y(s) + sY(s) + 2Y(s) = e^{-5s}$$

Step 2 Solve for $Y(s)$

$$(2s^2 + s + 2)\,Y(s) = e^{-s(5)}$$

$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2}$$

Step 3 Find $\mathcal{L}^{-1}\{Y(s)\}$

$$Y(s) = \frac{e^{-5s}}{2} \cdot \frac{1}{s^2 + \frac{1}{2}s + 2}$$

$$= \frac{e^{-5s}}{2} \cdot \frac{1}{s^2 + \frac{1}{2}s + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + 2}$$ ← complete the square

$$= \frac{e^{-5s}}{2} \cdot \frac{1}{\left(s^2 + \frac{1}{4}\right)^2 + \frac{15}{16}}$$ #9 table of transforms

$\to \frac{4}{\sqrt{15}} e^{-t/4} \sin\left(\frac{\sqrt{15}}{4}t\right)$

$$y(t) = \frac{2}{\sqrt{15}} \cdot u_5(t)\, e^{-(t-5)/4} \sin \frac{\sqrt{15}}{4}(t-5)$$

Convolution : $\mathcal{L}^{-1}\{F(s)\cdot G(s)\} \neq f_1(t)\cdot g(t)$ boo :(

→ Theorem: if $H(s) = F(s)\cdot G(s)$, and $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and $G(s) = \mathcal{L}^{-1}\{G(s)\}$ then

$$h(t) = \mathcal{L}^{-1}\{H(s)\}$$

$$h(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

↑ this is called the convolution of two functions.

→ Properties of the convolution: $(f*g) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$

1. $f*g = g*f$ (commutative)
2. $f*(g_1+g_2) = f*g_1 + f*g_2$ distributative
3. $f*(g*h) = (f*g)*h$ associative

Ex: Find $\mathcal{L}\{h(t)\}$ where

$$h(t) = \int_0^t \underbrace{(t-\tau)}_{f(t-\tau)}\underbrace{e^{\tau}}_{g(\tau)}\,d\tau$$

$= f*g$ where $f = t$ and $g = e^t$

$\Rightarrow \mathcal{L}\{h(t)\} = F(s)\cdot G(s)$

$f(t) = t$ $\quad$ $g(t) = e^t$

$F(s) = \frac{1}{s^2}$ $\quad$ $G(s) = \frac{1}{s-1}$

↑ via #3 $\quad$ ↑ via #2

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$$Y = G(s)\cdot\frac{1}{s^2+1} + \frac{s y_0}{s^2+1} + \frac{y_0'}{s^2+1}$$

$$y(t) = y_0'\sin(t) + y_0\cos(t) + \int_0^t \sin(t-\tau)g(\tau)d\tau$$