Math 234 / Katie Oliveras

Example

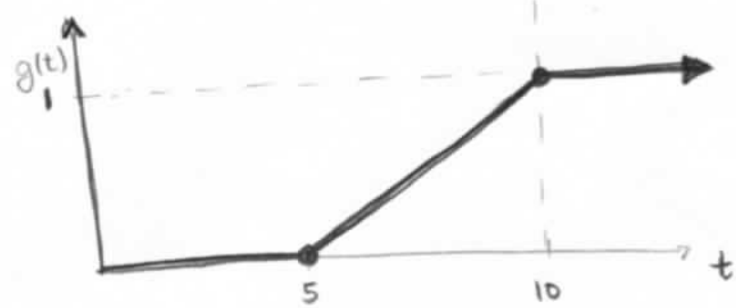
Solve the following ODE using Laplace transforms

$$y'' + 4y = g(t), \quad \text{where } y(0) = y'(0) = 0$$

and

$$g(t) = \begin{cases} 0 & 0 \le t < 5 \\ \frac{1}{5}\cdot(t-5) & 5 \le t < 10 \\ 1 & t \ge 10 \end{cases}$$

First, let's look at a graph of $g(t)$ and write it in terms of step functions.



* write as unit step function

For $t \in [0,5)$, there is no forcing, but when $t \in [5,10)$, there is linear forcing. Finally, when $t \ge 10$, the output is a constant value of 1.

$$\boxed{g(t) = u_5(t)\cdot\left(\frac{t-5}{5}\right) - u_{10}(t)\cdot\left(\frac{t-10}{5}\right)}$$

check: when $t < 5$,

$$g(t) = 0 - 0 \quad \checkmark$$

when $t \in [5,10)$

$$g(t) = 1\cdot\left(\frac{t-5}{5}\right) - 0 \quad \checkmark$$

when $t \in [10,\infty)$

$$g(t) = 1\cdot\left(\frac{t-5}{5}\right) - 1\cdot\left(\frac{t-10}{5}\right) = 1 \quad \checkmark$$

Now, let's solve the ode via Laplace transforms

$\boxed{\text{Step 1: take the } \mathcal{L}\{\ \} \text{ of the ode}}$

$$\mathcal{L}\{y'' + 4y'\} = \mathcal{L}\left\{u_5(t)\cdot\left(\frac{t-5}{5}\right) - u_{10}(t)\cdot\left(\frac{t-10}{5}\right)\right\}$$

Since $\mathcal{L}\{y\} = Y(s)$, $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$,

we have:

$$s^2Y(s) + 4Y(s) = \mathcal{L}\left\{u_5(t)\left(\frac{t-5}{5}\right) - u_{10}(t)\cdot\left(\frac{t-10}{5}\right)\right\}$$

$$\hookrightarrow (s^2+4)Y(s) = \mathcal{L}\left\{\tfrac{1}{5}u_5(t)\cdot(t-5)\right\} - \mathcal{L}\left\{\tfrac{1}{5}u_{10}(t)\cdot\left(\tfrac{t-10}{5}\right)\right\}$$

via #13

$$(s^2+4)Y(s) = \frac{1}{5}e^{-5s}\cdot\frac{1}{s^2} - \frac{1}{5}e^{-10s}\cdot\frac{1}{s^2}$$

$\boxed{\text{Step 2: solve for } Y(s)}$

$$(s^2+4)Y(s) = \frac{1}{5\cdot s^2}\left[e^{-5s} - e^{-10s}\right]$$

$$\hookrightarrow Y(s) = \frac{1}{5s^2(s^2+4)}\cdot\left[e^{-5s} - e^{-10s}\right]$$

Let $\underbrace{H(s) = \frac{1}{s^2(s^2+4)}}_{\text{for compactness}}$ $\rightarrow$ $\boxed{Y(s) = \frac{e^{-5s} - e^{-10s}}{5}\cdot H(s)}$

$\boxed{\text{Step 3: solve for } y(t) = \mathcal{L}^{-1}\{Y(s)\}}$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{e^{-5s} - e^{-10s}}{5} \cdot H(s)\right\} \quad \text{(expand and simplify)}$$

$$= \frac{1}{5}\left[\underbrace{\mathcal{L}^{-1}\{e^{-5s} H(s)\}}_{\downarrow} - \underbrace{\mathcal{L}^{-1}\{e^{-10s} H(s)\}}_{\downarrow}\right] \to \boxed{\text{via \#13}}$$

$$y(t) = \frac{1}{5}\left[u_5(t)\cdot h(t-5) - u_{10}(t)\cdot h(t-10)\right]$$

where $\to h(t) = \mathcal{L}^{-1}\{H(s)\}$

Thus, if we can find $\underline{h(t) = \mathcal{L}^{-1}\{H(s)\}}$, we can find $y(t)$.

$$H(s) = \frac{1}{s^2(s^2+4)} \to \text{repeated roots } \underline{s^2 = 0}$$

$\hookrightarrow$ use partial fractions!

$$H(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^2+4}$$

$$As(s^2+4) + B(s^2+4) + C(s^2) = 1$$

s^0: $4B = 1 \to \underline{B = 1/4}$ ✓

s^1: $4As = 0 \to \underline{A = 0}$ ✓

s^2: $B + C = 0 \to \underline{C = -1/4}$ ✓

$$\to H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4} \to H(s) = \frac{1}{4}\cdot\frac{1}{s^2} - \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{2}{s^2+4}$$

Using the table,

via #3 and #5

$$\boxed{\begin{aligned} h(t) &= \tfrac{1}{4}\cdot t - \tfrac{1}{8}\cdot\sin(2t) \\ &= \tfrac{1}{4}\cdot\left[t - \tfrac{1}{2}\sin(2t)\right] \end{aligned}}$$

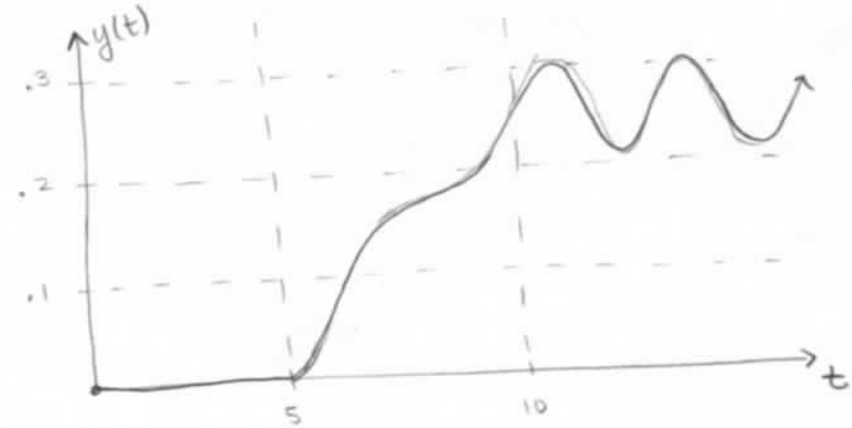
Thus, the final answer is...

$$y(t) = \frac{1}{5}\left[u_5(t)\cdot h(t-5) - u_{10}(t)\cdot h(t-10)\right]$$

$$y(t) = \frac{1}{20}\cdot\left[u_5(t)\cdot\left(t-5-\tfrac{1}{2}\sin(2(t-5))\right) - u_{10}(t)\cdot\left(t-10-\tfrac{1}{2}\sin(2(t-10))\right)\right]$$

$$\boxed{y(t) = \frac{1}{20}\left[u_5(t)\cdot\left[t-5-\tfrac{1}{2}\cdot\sin(2t-10)\right] - u_{10}(t)\cdot\left[t-10-\tfrac{1}{2}\sin(2t-20)\right]\right]}$$

The plot of the solution:



see book for better plot!