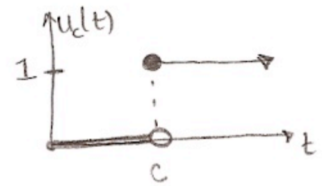
6.3 The unit step function

The <u>unit step</u> or <u>heavy-side function</u> gives us a way to easily deal with jump discontinuities.

For example: let $u_c(t)$ measure the current going through a circuit → turned on at $t=c$

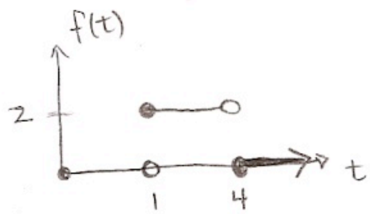


→ in normal piecewise notation

$$u_c(t) = \begin{cases} 0 & t<c \\ 1 & t \geq c \end{cases}$$

→ this is our def. for the unit step.

Ex. Write the function below using $u_c(t)$ notation



→ $\boxed{f(t) = 2 \cdot u_1(t) - 2 \cdot u_4(t)}$

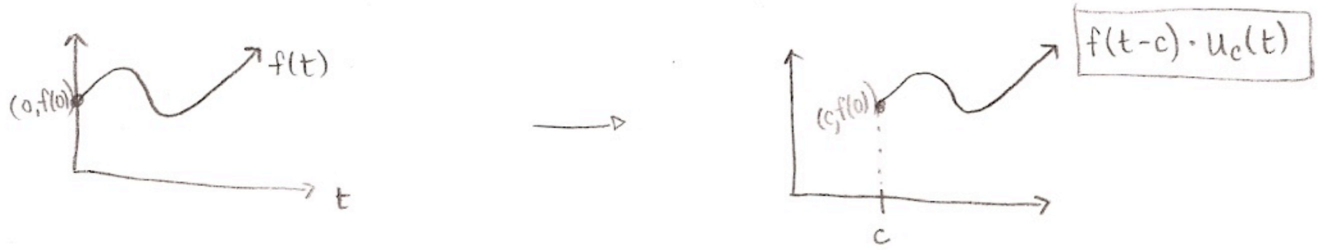
Ex: Find the Laplace transform of $u_c(t)$

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty u_c(t) e^{-st}\, dt = \int_0^c 0 \cdot e^{-st} dt + \int_c^\infty e^{-st}\, dt = \frac{1}{s} e^{-st}\Big|_c^\infty$$

$$= \lim_{A \to \infty} \frac{-1}{s}\left[e^{-As} - e^{-cs}\right] = \frac{e^{-cs}}{s}$$

→ $\boxed{\mathcal{L}\{u_c(t)\} = \frac{1}{s} e^{-cs}}$ $\boxed{s>0}$

Sometimes, we want to turn on a function at a particular time



*** Thm:** If $F(s) = \mathcal{L}\{f(t)\}$, then $\mathcal{L}\{f(t-c)\,u_c(t)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$

Likewise: $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)\cdot u_c(t)$

proof: substitution!

$$\mathcal{L}\{f(t-c)u_c(t)\} = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt$$
$$= \int_c^\infty e^{-st}f(t-c)\,dt$$

substitution

$\rightarrow$ let $\xi = t-c$ $\rightarrow$ $d\xi = dt$ $\rightarrow$ $t = \xi + c$

$$\rightarrow \int_0^\infty e^{-(\xi+c)s}f(\xi)\,d\xi = e^{-cs}\int_0^\infty e^{-s\xi}f(\xi)\,d\xi = \underline{e^{-cs}F(s)}$$

Ex: Find the Laplace transform of

$$f(t) = \begin{cases} t & 0 \le t < 2 \\ 2 & t \geq 2 \end{cases}$$

step 1: Note that we can rewrite $f(t)$ as $f(t) = t + g(t)$ where $g(t) = -u_2(t)\cdot(2-t)$

$\rightarrow$ $f(t) = t - u_2(t)(t-2)$ if $t<2 \rightarrow f(t) = t$

$t \geq 2$ $f(t) = t - 1\cdot(t-2)$

$= \cancel{t} - \cancel{t} + 2$ ✓

Step 2: Find $\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{u_2(t)\cdot(t-2)\}$ by #3 (n=1) and #13 on table.

$$= \frac{1}{s^2} - \frac{e^{-2s}}{s^2}$$

$$= \boxed{\frac{1-e^{-2s}}{s^2}}$$

Ex: Find $\mathcal{L}^{-1}\left\{ \dfrac{1+se^{s\pi/4}}{s^2+1} \right\}$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \right\} + \mathcal{L}^{-1}\left\{ \frac{se^{-s\pi/4}}{s^2+1} \right\} \longrightarrow e^{-s\pi/4} \cdot \frac{s}{s^2+1}$$

$$= \boxed{\sin(t) + u_{\pi/4}(t)\cdot\cos(t-\pi/4)}$$

Thm: $\mathcal{L}\{e^{ct}f(t)\} = F(s-c) \quad \to \mathcal{L}^{-1}\{F(s-c)\} = e^{ct}f(t)$

Ex. Find $\mathcal{L}^{-1}\left\{ \dfrac{1}{s^2-4s+5} \right\}$ → try completing the square in dent.

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s^2-4s+4-4+5} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{(s-2)^2+1} \right\}$$

$$= \boxed{e^{2t}\sin(t)}$$

how would you know to complete the square?

s^2-4s+5 has complex roots!

$$s = \frac{4 \pm \sqrt{16-20}}{2} \checkmark$$

Example (useful for HW)

The gamma function $\Gamma(p)$ is defined by

$$\Gamma(p+1) = \int_0^\infty e^{-x} x^p \, dx$$

where $\Gamma(p+1) = p\Gamma(p)$ → via integration by parts

Use the above to show that

$$\mathcal{L}\{t^{-1/2}\} = \Gamma(\tfrac{1}{2}) / s^{1/2}$$

$$\mathcal{L}\{t^{-1/2}\} = \Gamma(\tfrac{1}{2})/\sqrt{s}$$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2})$

$\frac{\sqrt{\pi}}{2} = \Gamma(\frac{3}{2})$

Note: if p is an integer...

$$\Gamma(p+1) = p \cdot \Gamma(p)$$
$$= p \cdot (p-1) \cdot \Gamma(p-1) \quad \ldots \quad \text{repeat}$$
$$= p \cdot (p-1) \cdot (p-2) \ldots (2) \cdot 1 \cdot \underbrace{\Gamma(1)}_{=1}$$
$$= p!$$

Also note that $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-x} x^{-1/2} \, dx = \sqrt{\pi}$

More generally...

$$\int e^{-st} t^n \, dt =$$

$u = st$ $\quad du = s\,dt$

$t = \frac{u}{s}$ $\quad dt = \frac{1}{s} du$

$$= \int e^{-u} \left(\frac{u}{s}\right)^n \cdot \frac{1}{s} du$$

$$= \frac{1}{s^{n+1}} \underbrace{\int e^{-u} u^n \, du}_{\Gamma(n+1)}$$

if n is an integer

$$= \boxed{\frac{1}{s^{n+1}} \cdot n!}$$