Solving differential eqns w/ Laplace transform

① Find the Laplace transform of $f'(t)$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

→ integrate by parts

$u = e^{-st}$ $\qquad dv = f'(t)\,dt$

$du = -se^{-st}$ $\qquad v = f(t)$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{A\to\infty} \underbrace{e^{-sA} f(A)}_{\to 0} - f(0) + s\cdot\mathcal{L}\{f(t)\}$$

assume this works!

$$\rightarrow \boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$$

② Find the Laplace transform of $f''(t)$

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{(f'(t))'\}$$ use the above ...

$$= s\cdot\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\cdot\left(s\,\mathcal{L}\{f(t)\} - f(0)\right) - f'(0)$$

$$\boxed{\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)}$$

In general

$$\rightarrow \boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)}$$

Simple Example

$$y'' - y' - 2y = 0 \qquad y(0) = 1 \quad y'(0) = 0$$

Step 1 take Laplace transform noting that

$$\mathcal{L}\{y''\} = s^2 Y(s) - s y(0) - y'(0) = s^2 Y(s) - s - 0$$

reverse order

$$\mathcal{L}\{y'\} = s Y(s) - y(0) = s Y(s) - 1$$

$$\mathcal{L}\{y\} = Y(s)$$

$$s^2 Y(s) - s - (s Y(s) - 1) - 2Y(s) = 0$$

$$s^2 Y(s) - s - s Y(s) + 1 - 2Y(s) = 0$$

Step 2 solve algebraically for $Y(s)$

$$(s^2 - s - 2) Y(s) - s + 1 = 0$$

$$(s^2 - s - 2) Y(s) = s - 1$$

$$\boxed{Y(s) = \frac{s-1}{s^2 - s - 2}}$$

What we have now found is that the laplace transform of the solution is given by

$$\mathcal{L}\{y(t)\} = \frac{s-1}{s^2 - s - 2}$$

If we can go from $Y(s) \longrightarrow y(t)$ we have found the solution of the IVP.

Step3 Find inverse Laplace transform.

Mathematica: Apart[1/((1+s)(s+s))] → Wolfram Alpha

We have found the Laplace transform of our solution!

to find the solution $y(t)$ we need to go back or take the inverse Laplace transform.

→ the inverse Laplace transform is beyond the scope of this class. It requires complex analysis

→ Most people really just use a table of transforms. (pass out table)

→ Typically have to use partial fractions to solve!

$$Y(s) = \frac{s-1}{(s^2-s-2)} = \frac{s-1}{(s-2)(s+1)}$$

$$\rightarrow Y(s) = \frac{A}{s-2} + \frac{B}{s+1}$$

$$\frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$ multiply by $(s-2)(s+1)$

$$\rightarrow s-1 = A(s+1) + B(s-2)$$ collect powers of s

$$s-1 = As + A + Bs - 2B$$

s^0: $-1 = A - 2B$

s^1: $1 = A + B$ → $A = 1 - B$

$-1 = 1 - B - 2B$ → $-1 = 1 - 3B$

$-2 = -3B$ → $B = \frac{2}{3}$ $A = \frac{1}{3}$

$$\rightarrow \boxed{Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}}$$ → table $\boxed{\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} = y(t)}$

The inverse transform will something we have to calculate a lot!

⇒ Partial fractions is something we will have to use a lot!

We just worked an example where the roots in the denominator were real and unique. What if they are repeated?

Ex. Find the inverse Laplace transform of

$$Y(s) = \frac{s+1}{(s+2)^2}$$

partial fractions

$$Y(s) = \frac{s+1}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$$

multiply by $(s+2)^2$

$$\rightarrow A(s+2) + B = s+1$$

s^0: $2A + B = 1$

s^1: $A = 1$ $\rightarrow$ $\underline{A=1}$ $\rightarrow$ $2+B=1$ $\rightarrow$ $\underline{B=-1}$

$$Y(s) = \frac{1}{s+2} - \frac{1}{(s+2)^2}$$ $\rightarrow$ via #2 + #11 on table

$$\boxed{y(t) = e^{-2t} - te^{-2t}}$$