The Laplace Transforms

Perhaps one of the most important topics you will learn in this class for applications is the Laplace transforms.

Laplace transforms will turn differential equations into algebra. Even when the forcing functions are discontinuous!

→ Let's start by defining the Laplace transform.

The Laplace transform is an integral Transformation (one of many)

We use the notation $\mathcal{L}\{f(t)\}$ to denote the Laplace transform of $f(t)$.

Definition:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

→ the Laplace transform maps from the time domain → s-domain

Since the Laplace transform is an improper integral we evaluate it as a limit

$$\mathcal{L}\{f(t)\} = \lim_{A\to\infty} \int_0^A e^{-st}\cdot f(t)\,dt$$

→ Now, this integral can possibly diverge!

Let's consider the Laplace transform by splitting..

$$F(s) = \lim_{A\to\infty} \int_0^A e^{-st}\cdot f(t)\,dt = \underbrace{\int_0^M e^{-st} f(t)\,dt}_{\text{this is a finite \#.}} + \lim_{A\to\infty} \int_M^A e^{-st} f(t)\,dt$$

If $|f(t)| < Ke^{at}$ when $t \geq M$, then

$$|e^{-st} f(t)| \leq Ke^{-st}e^{at} = Ke^{(a-s)t}$$

The interval will converge for $\boxed{a-s<0}$

Example

Find $\mathcal{L}\{f(t)\}$ for each of the following:

① $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{a\to\infty} -\frac{1}{s}e^{-st}\Big|_0^a$

$= \lim_{a\to\infty} -\frac{1}{s}\left[e^{-at} - e^0\right]$

$= \boxed{\frac{1}{s}}, \quad \underline{s>0}$

② $f(t) = e^{at}$

$\mathcal{L}\{f(t)\} = \lim_{\alpha\to\infty} \int_0^\alpha e^{-st} e^{at}\,dt$

$= \lim_{\alpha\to\infty} \left[\frac{1}{a-s}\left(e^{(a-s)t}\right)\right]_0^\alpha$

$= \boxed{+\frac{1}{s-a}}, \quad s>a$

③ $F(s) = \lim_{A\to\infty} \int_0^A e^{-st}\sin(at)\,dt$

$\Rightarrow$ integrate by parts.

$u = e^{-st} \qquad dv = \sin(at)$

$du = -se^{-st} \qquad \underline{v = \frac{-1}{a}\cos(at)}$

$F(s) = -\frac{1}{a}e^{-st}\cos(at)\Big|_0^A - \lim_{A\to\infty}\frac{s}{a}\int_0^A e^{-st}\cos(at)\,dt$

show more steps

$F(s) = -\frac{1}{a}e^{-st}\cos(at)\Big|_0^A - \frac{s^2}{a^2}\int_0^A e^{-st}\sin(at)\,dt$

$F(s) = \frac{1}{a} - F(s)\frac{s^2}{a^2}$

$a^2\left(1+\frac{s^2}{a^2}\right)F(s) = a^2\frac{1}{a} \longrightarrow \boxed{F(s) = \frac{a}{a^2+s^2}}$

---

discontinuous example

$f(t) = \begin{cases} -1 & t<4 \\ 1 & t\geq 4 \end{cases} \longrightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$

$= \int_0^4 e^{-st}(-1)\,dt + \int_4^\infty e^{-st}\,dt$

etc...