Math 234 - Lecture Notes†

Introduction to Differential Equations

Overview

In today's lecture, we will discuss some of the basic terminology we will use for the remainder of the quarter. We will also discuss a method to determine the long term behavior of solutions to differential equations without needing to find the solution.

Algebraic equations

An algebraic equation is an equation between an unknown quantity x and functions of this quantity x. It may be written in the form

$$F(x) = 0,$$

such as $2x^2 + x - 3 = 0$. In this case, the function $F(x)$ is simply the expression $F(x) = 2x^2 + x - 3$.

If there are multiple variables, say x and y, then the equation is of the general form

$$F(x, y) = 0,$$

where F is a vector. As an example,

$$\begin{aligned} x^2 - y^2 &= 2, \\ x + y &= \cos(x). \end{aligned}$$

In the above example, the vector function $F(x, y)$ is given by

$$F(x, y) = \begin{bmatrix} x^2 - y^2 - 2 \\ x + y - \cos(x) \end{bmatrix}$$

Solutions of algebraic equations

If you are given an algebraic equation (or a set of algebraic equations) to solve, you would need to find a number or a set of numbers that satisfy $F(x) = 0$ (or $F(x, y) = 0$, etc). Often, finding solutions to algebraic equations can be difficult; especially when the algebraic equations contain transcendental functions such as sin and cos. However, if you are given a numeric value, it is easy to check if it is a solution.

Example

Verify that $x = 4$ is a solution of $x^2 - 2x - 8 = 0$.

Solution

Plug $x = 4$ into the equation:

$$4^2 - 2 \cdot 4 - 8 = 0 \quad \Longrightarrow \quad 0 = 0 \checkmark$$

Thus indeed, $x = 4$ solves the equation.

Note that checking that a given number solves the algebraic equation requires no more effort than plugging the proposed solution into the equation. This is a lot easier than actually finding a solution.

Differential equations

A differential equation is a relationship between a function and its derivatives. The difference between an algebraic equation and a differential equation is that the solution to a differential equation is a function instead of a number.

An example of a differential equation is $y' + 2y = 1$, where y is a function of x ($y = y(x)$). In order to solve this equation, we need to find **all** functions that satisfy it.

Example

Verify that $y = \frac{1}{2} + ce^{-2x}$ is a solution of the differential equation $y' + 2y = 1$, with c being a constant.

Solution

To check, we simply plug the function $y = \frac{1}{2} + ce^{-2x}$ into the differential equation. In order to plug in $y(x)$ into the differential equation, we need to first calculate y'.

$$y' = 0 + c \cdot (-2)e^{-2x} \quad y' = -2ce^{-2x}.$$

Plugging this into the differential equation, we have

$$\begin{aligned} -2ce^{-2x} + 2\left(\frac{1}{2} + ce^{-2x}\right) &\stackrel{?}{=} 1 \\ 1 &\stackrel{?}{=} 1 \ \checkmark. \end{aligned}$$

Therefore, $y(x) = \frac{1}{2} + ce^{-2x}$ is **a** solution of the differential equation $y' + 2y = 1$.

We should not be surprised there is an arbitrary constant in the solution. The differential equation contains one derivative. To get rid of it, you will need to integrate at some point. This integration will result in an integration constant.

This is pretty good: although we don't know yet how to solve differential equations, we already know how to verify that something is a solution. Note that this verification requires us to take derivatives. That's okay: taking derivatives is mechanical. There's a set of rules, and if we follow these rules, we're doing fine. Since solving differential equations requires us to get rid of derivatives, you might justifiably think that integration enters into it. But integration is a lot harder than differentiation: there are some rules, but often there are tricks to be used. Even more often, integrals cannot be explicitly done. So be it. In summary:

CHECKING = PLUGGING IN!

General Definitions / Terminology

Now, we will introduce some basic terminology that we will use throughout the course.

Definition: Order

The **order** of a differential equation is the order of the highest derivative appearing in it.

Example

For each of the following differential equations, determine the order:
(a) $y' + 2y = 1$,
(b) $y''' = y'' - y + \sin(x)$,
(c) $\frac{dx}{dt} + \frac{d^2x}{dt^2}\frac{dx}{dt} + x^2 = 2$

Solution

(a) first order
(b) third order
(c) second order

Definition: Ordinary vs. Partial Differential Equation

If a differential equation contains derivatives with respect to only one variable, it is called an **ordinary differential equation**. Otherwise it is called a partial differential equation.

Example

Which of the following equations is an ordinary differential equation:
(a) $\frac{du}{dt} = 2\frac{u}{x}$,
(b) $u_x = \sin(x)$

Solution

Equation (a) is a partial differential equation, and equation (b) is a ordinary differential equation.

Definition: Linear vs. Nonlinear Differential Equation

An equation is called **linear** if the unknowns in it appear in a linear way: they do not multiply each other or themselves, and they do not appear as arguments of nonlinear functions.

Example

Determine if each of the following ODEs is linear or nonlinear:
(a) $y' = xy^3$,
(b) $u_x = u\sin(x)$

Solution

(a) nonlinear ordinary differential equation because of the y^3 term,
(b) is a linear ordinary differential equation.

Example

Check that $y = c_1\cos(x) + c_2\sin(x)$ is a solution of $y'' + y = 0$. Note that this is a linear second-order equation.

Solution

Plugging in gives:

$$\begin{aligned}
& & y' &= -c_1\sin(x) + c_2\cos(x) \\
& & y'' &= -c_1\cos(x) - c_2\sin(x) \\
\Rightarrow & & -c_1\cos(x) - c_2\sin(x) + (c_1\cos(x) + c_2\sin(x)) &\overset{?}{=} 0 \\
\Rightarrow & & 0 &\overset{!}{=} 0.
\end{aligned}$$

Note that the solution of this second-order problem depends on two arbitrary constants c_1 and c_2. This is to be expected: in order to get rid of two derivatives, you expect to have to integrate twice, resulting in two integration constants.

Definition: Initial Value Problem (IVP)

An **initial-value problem** is a differential equation together with some algebraic conditions which allow you to determine the arbitrary constants. In general, if you want to determine all arbitrary constants, you need to specify as many **initial conditions** as the order of the equation.

Example

Show that $y = \frac{1}{2} + \frac{1}{2}e^{-2x}$ satisfies the IVP

$$\begin{aligned} y' + 2y &= 1, \\ y(0) &= 1. \end{aligned}$$

Solution

We have already verified that the given y satisfies the differential equation, so it is left to check that it satisfies the initial condition. At $x = 0$, $y = 1/2 + 1/2 = 1$, so the initial condition is indeed satisfied. Thus,

$y = \frac{1}{2} + \frac{1}{2}e^{-2x}$ is a solution to the above IVP.

Example

Find c_1, c_2 so that $y = c_1 \cos(x) + c_2 \sin(x)$ satisfies the initial-value problem

$$\begin{aligned} y'' + y &= 0 \\ y(0) = 1, y'(0) &= 0 \end{aligned}$$

Solution

We already know that the given $y(x)$ satisfies the differential equation for all choices of c_1 and c_2.

We have $y(0) = c_1$, so that from the first initial condition it follows that $c_1 = 1$. Next, $y' = -c_1 \sin(x) + c_2 \cos(x)$, so that $y'(0) = c_2$.

It follows from the second initial condition that $c_2 = 0$, so that the solution to the initial-value problem is $y = \cos(x)$.

When checking solutions to IVPs, it is important to check that the solution satisfies both the differential equation and the algebraic constraints.

Guessing solutions

Our main method for solving differential equations in this course will be: (drum roll...)

$$\boxed{\text{GUESSING!}}$$

Often we will guess the form of a solution. A suitable form for the solution will depend on a few parameters. We will adjust the parameters to make the solution work.

Example

Consider the differential equation $y'' + 3y' - 4y = 0$. Find all functions $y(x)$ such that when you take a linear combination of $y(x)$ and two of its derivatives, you get zero. In other words, find a function $y(x)$ whose derivatives are very similar to it.

Solution

One such function is $y = e^{ax}$, where a is a constant. Let's check to see if this works.

$$\begin{aligned} && y =& e^{ax} \\ \Rightarrow && y' =& ae^{ax} \\ \Rightarrow && y'' =& a^2e^{ax} \\ \Rightarrow && y'' + 3y' - 4y =& a^2e^{ax} + 3ae^{ax} - 4e^{ax} \\ && =& (a^2 + 3a - 4)e^{ax}. \end{aligned}$$

So, this does not work... unless $a^2 + 3a - 4 = 0$, *ie*, $a = 1$ or $a = -4$. In other words,

$$y_1 = e^x, \quad y_2 = e^{-4x}$$

are both solutions. By guessing the functional form of a solution, we reduced the problem of solving a differential equation to the problem of solving an algebraic equation. This is definitely progress! We don't have all solutions yet, as the general solution should depend on two arbitrary constants. In the lectures on second-order equations we will learn how to use the two solutions we just found to construct the general solution.

Direction fields

For any first-order differential equation

$$y' = f(x, y),$$

we can get a graphical idea of what the solutions look like, even if we can't solve the equations. At any point (x_0, y_0) in the (x, y)-plane, the equation tells us what the rate of change of the solution through this point is. So, if we happen to find ourselves at this point (perhaps the initial condition put us there), the equation tells us how to move on from the point where we are.

The collection of all arrows through all points is called the **direction field** of the differential equation. The rate of change at any point gives the tangent vector to the solution curve through this point, allowing us to draw the tangent vector to the curve $y = y(x)$, which solves the differential equation, even if we cannot determine the form of this solution.

To summarize, you can draw a direction field by following these steps:

1. Choose a point in the x-y plane (x_0, y_0), and evaluate y' at this point.
2. At the point (x_0, y_0), draw a small arrow with the slope found in part (1).
3. Repeat this process for a lot of points (x_0, y_0). The plot that you obtain will give you a good idea of how solutions behave. *NOTE: You can use the dfield applet found at* http://math.rice.edu/ dfield/dfpp.html *to automatically generate this plot.*

For example, consider the differential equation

$$y' = y^2 - x$$

By choosing the following (x, y) points, we can find the value of dy/dx at those values to create the following table:

x	y	dy/dx
0	0	0
1	0	-1
0	1	1
1	1	1
2	2	2
$\vdots$	$\vdots$	$\vdots$

Thus, to find out what $y(x)$ looks like:

$$\boxed{\text{FOLLOW THE ARROWS!}}$$

The final direction field for $y' = y^2 - x$ (along with some solution curves) is shown in Figure 1

As you may see from this direction fields, they may often be used to understand the long-time behavior of solutions, which in many applications is all we care about.

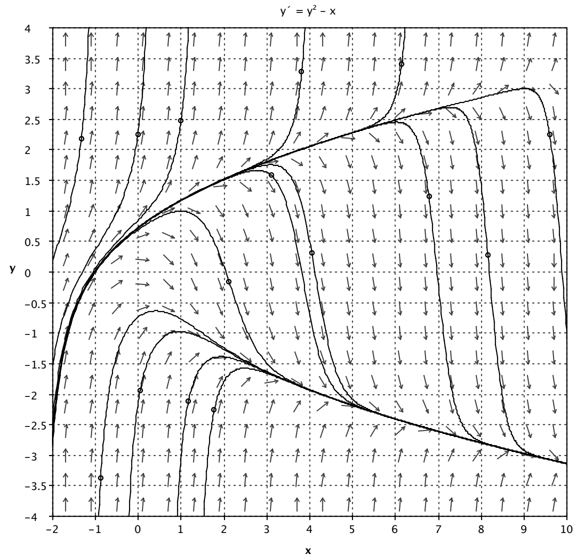

Figure 1: The direction field for the equation $y' = y^2 - x$, together with some inferred solution curves.

Example

Using the direction field, determine the long term behavior of the solution to the initial value problem

$$y' = \cos(y), \qquad y(0) = 0$$

Solution

Looking at the direction field (see Figure 2), it appears that $y(x) \to \frac{\pi}{2}$ as $x \to \infty$.

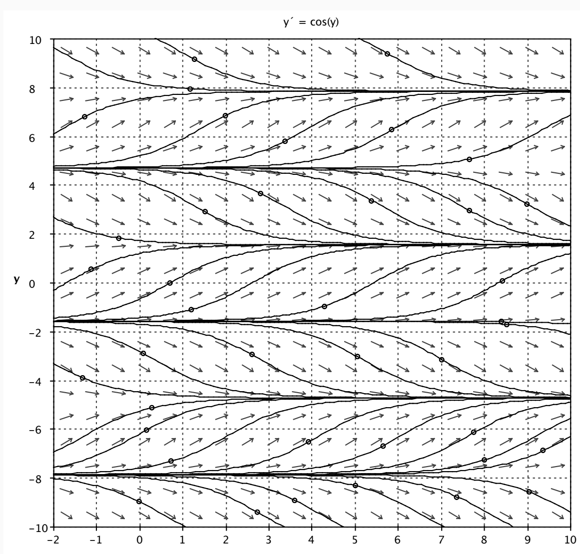

Figure 2: The direction field for the equation $y' = \cos(y)$, together with some inferred solution curves.

Lagniappe:

Notice that the limit as $x \to \infty$ depends on $y(0)$. If we look at the pattern on the vertical axis, we can actually write down the limit as $x \to \infty$ for all initial conditions!

Let $y(0) = y_0$, where y_0 is some arbitrary constant. We can write the following:

$$\lim_{x\to\infty} y(x) = \begin{cases} \dfrac{(2n+3)\pi}{2} & \dfrac{(2n+1)\pi}{2} < y_0 < \dfrac{(2n+5)\pi}{2}, \quad n \in \mathbb{Z} \\ \\ \dfrac{(2n+1)\pi}{2} & y_0 = \dfrac{(2n+1)\pi}{2}, \quad n \in \mathbb{Z} \end{cases}$$

where $\mathbb{Z}$ represents all integers $(\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots)$.

Of course, drawing all the tiny vectors in a direction field is a lot of work. It's also very boring work. In other words, it is the kind of work that a computer is very good at. On the course webpage you will find a link to a Java applet by John Polking and others to draw direction fields. It is available at `http://math.rice.edu/~dfield/dfpp.html`. The applet also allows you to draw in solution curves by clicking on the point through which you wish to draw a curve.

† These lecture notes are based on those of Dr. Bernard Deconinck at the University of Washington. They have been modified to fit this class.