Math 234 - Lecture Notes†

First Order Separable Differential Equations

Overview

Now will will begin with the process of learning how to solve differential equations. We will learn different techniques for different types of differential equations. Generally, linear differential equations are among the easiest to solve. In fact, almost all differential equations and solution techniques we will consider in this course are linear. However, this lecture is one of the few exceptions.

Consider a differential equation of the form

$$\frac{dy}{dx} = f(x, y).$$

This is the most general form of a first-order differential equation. If $f(x, y)$ can be written as the product of a function of x and a function of y, we can solve the equation. Such an equation is called **separable**. It is of the form

$$\boxed{\frac{dy}{dx} = g(x)\ h(y).}$$

Then

$$\frac{1}{h(y)} y'(x) = g(x).$$

Integrating both sides with respect to x gives

$$\int \frac{1}{h(y)} y'(x) dx = \int g(x) dx.$$

The integral on the left may be rewritten as

$$\int \frac{1}{h(y)} dy,$$

and we get

$$\boxed{\int \frac{1}{h(y)} dy = \int g(x) dx + c,}$$

where we have written the constant of integration explicitly, so that we do not forget it.

Now the problem has been reduced to a calculus problem, and the differential equation has been solved. Note that even if we cannot do the integral, we consider the differential equation solved because there are no more derivatives in the problem.

Even if we can do the integral, it is unlikely that we can solve the resulting equation for y. So be it. If we can solve for y as a function of x we say that we have found an **explicit solution**. If not, we say we have an **implicit** solution.

EXAMPLE

Consider the initial-value problem

$$y' = -6xy, \qquad y(0) = -4.$$

SOLUTION

From the first equation, we obtain

$$\begin{aligned} & \frac{1}{y}dy = -\,6x dx \\ \Rightarrow \quad & \int \frac{1}{y}dy = -\,6\int x dx + c \\ \Rightarrow \quad & \ln(y) = -3x^2 + c. \end{aligned}$$

This is an implicit solution of the differential equation. In this case, we can solve for y:

$$\begin{aligned} y &= e^{-3x^2+c} \\ &= e^{-3x^2}e^c \\ &= Ce^{-3x^2}, \end{aligned}$$

where we have set $C = e^c$, an arbitrary constant. Now we may use the initial condition:

$$-4 = Ce^0 \quad \Rightarrow \quad C = -4,$$

and the explicit solution is

$$y = -4e^{-3x^2}.$$

Example

Consider the differential equation

$$y' = \frac{3x^2 + 4x + 2}{2(y-1)}$$

Solution

$$\begin{aligned}(2y-2)dy &= (3x^2+4x+2)dx \\ \int (2y-2)dy &= \int (3x^2+4x+2)dx + c \\ y^2 - 2y &= x^3 + 2x^2 + 2x + c.\end{aligned}$$

This is an implicit solution to the differential equation. In this case, we can actually write down the explicit solution. This amounts to solving the above quadratic equation for y explicitly, which may be done easily by completing the square:

$$\begin{aligned}y^2 - 2y + 1 &= x^3 + 2x^2 + 2x + c + 1 \\ \Rightarrow (y-1)^2 &= x^3 + 2x^2 + 2x + c + 1 \\ \Rightarrow y - 1 &= \pm\sqrt{x^3 + 2x^2 + 2x + c + 1} \\ \Rightarrow y &= 1 \pm \sqrt{x^3 + 2x^2 + 2x + c + 1}.\end{aligned}$$

This is the explicit solution of the differential equation. As you may deduce from this example, in many cases it is much harder to find an explicit solution than an implicit solution.

The solution curves for this example are plotted in Figure 1. On the line $y = 1$, the solution curves have a vertical tangent. All curves above the line $y = 1$ correspond to the $+\sqrt{\ }$ for the explicit solution, whereas all curves below $y = 1$ correspond to the $-\sqrt{\ }$ for the explicit solution.

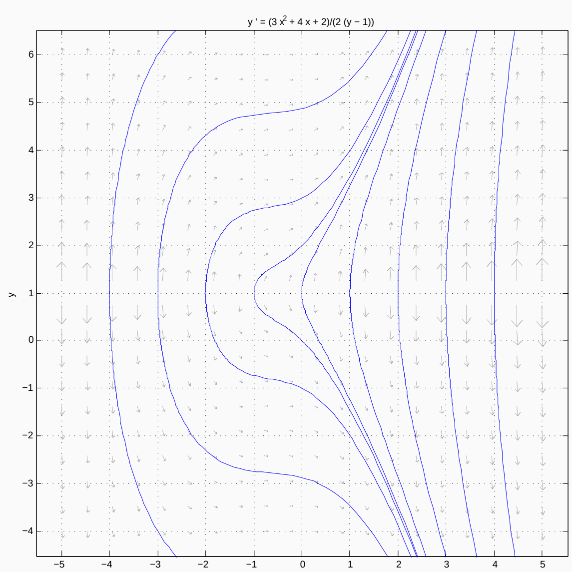

Figure 1: Direction field for $y' = \dfrac{3x^2 + 4x + 2}{2(y-1)}$

Example

Suppose we have to solve the initial-value problem

$$y' = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1$$

Solution

Using the implicit solution, we get

$$1 + 2 = 0 + c \quad \Rightarrow \quad c = 3,$$

and thus

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3,$$

which tells us which solution curve to use, but not which part of it.
Using the explicit solution we obtain

$$\begin{aligned} -1 &= 1 \pm \sqrt{c+1} \\ \Rightarrow \quad -2 &= \pm\sqrt{c+1}. \end{aligned}$$

Independent of what value we find for c, this equality can only hold if we use the $-$ sign. Proceding with this choice:

$$-2 = -\sqrt{c+1} \quad \Rightarrow \quad \sqrt{c+1} = 2 \quad \Rightarrow \quad c = 3,$$

giving the explicit solution

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

The explicit solution conveys which solution curve has to be used, and also which part of it is found.

Example

This example demonstrates that it is not always possible to find an explicit solution, even if we can solve the differential equation. Consider the initial-value problem

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}, \qquad y(0) = 1$$

Solution

We obtain

$$\begin{aligned} & \frac{1+2y^2}{y}dy = \cos x dx \\ \Rightarrow \quad & \int \frac{1+2y^2}{y}dy = \int \cos x dx + c \\ \Rightarrow \quad & \int \left(\frac{1}{y} + 2y\right) dy = \sin x + c \\ \Rightarrow \quad & \ln y + y^2 = \sin x + c. \end{aligned}$$

This is the implicit solution of the differential equation. It is not possible to solve this equation for y as a function of x, thus no explicit solution can be found. Nevertheless, we can still solve the initial-value problem. From the initial condition:

$$\ln 1 + 1^2 = \sin 0 + c \quad \Rightarrow \quad c = 1,$$

so that the implicit solution of the initial-value problem is

$$\ln y + y^2 = \sin x + 1.$$

In what follows, I want to demonstrate that when solving separable equations, you have to be careful when you divide. Consider the following

Example

Consider the initial value problem

$$y' = y^2, \qquad y(0) = 0$$

Find the solution.

Solution

Proceeding without caution:

$$\begin{aligned}
\Rightarrow \quad & \frac{1}{y^2}dy = dx \\
\Rightarrow \quad & \int \frac{1}{y^2}dy = \int dx + c \\
\Rightarrow \quad & -\frac{1}{y} = x + c \\
\Rightarrow \quad & y = \frac{1}{x+c}.
\end{aligned}$$

Now we use the initial condition, which leads to $0 = -1/c$, which cannot be solved for c! The problem occured right at the beginning, where we divided by y^2, which we may only do if $y \neq 0$. As it turns out, for the given differential equation, $y = 0$ is exactly what we need.

In general, whenever we divide by a function of y, we need to check what happens when the denominator of this function is zero. Let's try this example again, being more careful.

Proceeding with caution, we need to split the solution in two cases:

Case $y = 0$. In this case we cannot divide by y^2. Let's see if $y = 0$ is a solution of the differential equation: plugging in gives

$$0 \stackrel{!}{=} 0,$$

thus $y = 0$ is a solution! Even better, it is the solution that satisfies the initial condition. Thus, in summary, the solution of the initial value problem is $y = 0$.

Case $y \neq 0$. If different initial conditions are given, we have

$$\begin{aligned}
\frac{1}{y^2}dy &= dx \\
\int \frac{1}{y^2}dy &= \int dx + c \\
-\frac{1}{y} &= x + c \\
y &= \frac{-1}{x+c}.
\end{aligned}$$

† These lecture notes are based on those of Dr. Bernard Deconinck at the University of Washington. They have been modified to fit this class.