# MATH 234 - LECTURE NOTES†

# LINEAR FIRST ORDER DIFFERENTIAL EQUATIONS

## OVERVIEW

**EXAMPLE**

For each of the following first order differential equations, determine whether it is linear or nonlinear and whether it is separable.

(a) $\dfrac{dy}{dx} = y^2 + 3y$

(b) $y\dfrac{dy}{dx} = x$

(c) $\dfrac{dy}{dx} = 2 - y$

(d) $\dfrac{dy}{dx} = x - y$

**SOLUTION**

(a) $\dfrac{dy}{dx} = y^2 + 3y$ nonlinear and separable

(b) $y\dfrac{dy}{dx} = x$ nonlinear and separable

(c) $\dfrac{dy}{dx} = 2 - y$ linear and separable

(d) $\dfrac{dy}{dx} = x - y$ linear and NOT separable

So far, we know how to solve (a) - (c) in the example above (try for practice). However, even though (d) is linear, we do not know how to solve it just yet. In today's lecture, we will discuss a general method for solving linear first order differential equations even if they're not separable!

## SOLVING LINEAR FIRST ORDER DIFFERENTIAL EQUATIONS

Consider the following differential equation

$$y' = f(x, y)$$

This is the most general first order differential equation. Since we are only discussing *linear* differential equations in this lecture, we can write the form for the most general linear first order ODE as

$$y' + p(x)y = g(x) \tag{1}$$

where $p(x)$ and $g(x)$ are well-behaved functions of $x$[1]. We would like to develop a general method to solve (1).

With separable ODEs, all we needed to do was separate and integrate. We would like to be able to directly integrate (1); however, given its current form, we can't. Is there something we can do to make this happen? More specifically, can we satisfy the following goal?

**Goal:** Can we find some function of $x$ (say $\mu(x)$) so that if we multiply (1) by this special function $\mu(x)$, we can rewrite the differential equation as

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)g(x)$$

so that we can directly integrate the equation in order to solve for $y$.

Let's discuss *how* to find such a function, and how we can use it to solve for $x$.

Let's start by considering the expression $\frac{d}{dx}\left(\mu(x)y(x)\right)$. If we use the product rule to simplify the derivative, we get

$$\frac{d}{dx}\left(\mu(x)y(x)\right) = \mu' y + \mu y'. \tag{2}$$

Now, if we multiply the original ode (given in (1)) by the function $\mu(x)$ we have an expression of the form

$$\mu y' + \mu p(x)y = \mu(x)g(x). \tag{3}$$

If we compare the right hand side of equation (2) and the left hand side of (3), we find:

$$\begin{aligned} \mu y' + \mu p(x)y &= \mu' y + \mu y' \\ \cancel{\mu y'} + \mu p(x)y &= \mu' y + \cancel{\mu y'} \\ \mu p(x)y &= \mu' y \\ \mu p(x) &= \mu' \end{aligned}$$

Thus, if we can find some function $\mu(x)$ such that $\mu' = p(x)\mu$ then we can re-write (1) as

$$\frac{d}{dx}\left(\mu y\right) = \mu(x)g(x)$$

which is now a separable ode! If we integrate both sides of the differential equation with respect to $x$, we find

[1] The behavior of the functions $p(x)$ and $g(x)$ have a lot to do with the existence and uniqueness of solutions to differential equations. Will will discuss the importance of the functions in class soon.

$$\int \frac{d}{dx}\left(\mu(x)y(x)\right)\ dx = \int \mu(x)g(x)dx$$

If we integrate the above expression, we find that

$$\mu(x)y(x) = \int \mu(x)g(x)\ dx + c,$$

where we have explicitly represented the constant of integration $c$. If we solve the above expression for $y(x)$, we have

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)g(x)\ dx + c\right]$$

Now, to find $\mu(x)$, all we need to do is solve the equation

$$\mu' = p(x)\mu$$

This is pretty straight forward since we have a separable ODE for the function $\mu(x)$ in terms of the known function $p(x)$. In other words

$$\ln \mu = \int p(x)\ dx + c \quad \Rightarrow \quad \mu(x) = Ae^{\int p(x)\ dx}$$

Since we only need *a* function that works, we can choose $A = 1$. What does this all mean? It means, that $y(x)$ given above is the solution to the general first order linear ode given in (1). This allows us to come up with our first formula:

FORMULA

The general solution of a first order linear ODE of the form

$$y' + p(x)y = g(x)$$

is given by

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)g(x)\ dx + c\right]$$

where $\mu$ is given by

$$\mu(x) = e^{\int p(x)\ dx}$$

Typically, we refer to the function $\mu(x)$ as the integrating factor since multiplying the original ODE by $\mu(x)$ turns the left hand side into a perfect derivative and hence, we can integrate!

However, in general, we shouldn't use the formula to solve the differential equation. Instead, we should proceed by using the following steps to solve linear, first-order differential equations.

## Solving First-Order Linear ODES

1. Write the linear, first-order ODE in the standard form $y' + p(x)y = g(x)$.
2. Calculate the integrating function $\mu(x) = \exp\left[\int p(x)\,dx\right]$.
3. Multiply the ODE in standard form by $\mu(x)$ and write it in the form $\frac{d}{dx}[\mu(x)y(x)] = \mu(x)g(x)$.
4. Integrate this equation to obtain $\mu(x)y(x) = \int \mu(x)g(x)\,dx + c$.
5. Solve for $y(x)$ to obtain the general solution in explicit form.

## Example

Find the solution to the initial value problem

$$xy' + 2y = x^2 - x + 1, \qquad y(1) = \frac{1}{2}, \quad x > 0$$

## Solution

1. First, we must write the differential equation in standard form:

$$y' + \frac{2}{x}y = \frac{x^2 - x + 1}{x}$$

   This implies that

$$p(x) = \frac{2}{x}, \quad \text{and} \quad g(x) = \frac{x^2 - x + 1}{x}$$

2. Now we must find the integrating factor $\mu(x)$. For this problem, we find that

$$\mu(x) = \exp\left[\int \frac{2}{x}\,dx\right] = \exp\left[2\ln(x)\right] = x^2$$

3. Now, we can multiply the ODE in standard form by the $\mu(x)$ that we found above to get

$$\frac{d}{dx}\left[x^2 y(x)\right] = x^2 \cdot \frac{x^2 - x + 1}{x}$$

4. Integrating both sides with respect to $x$, we find

$$x^2 y(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + c.$$

5. Solving for $y(x)$, we find that

$$y(x) = \frac{1}{4}x^2 + \frac{1}{3}x + \frac{1}{2} + \frac{c}{x^2}.$$

Using the initial conditions and simplifying, we have

$$\boxed{y(x) = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{1}{12x^2}}$$