MATH 234 - LECTURE NOTES†

EXISTENCE AND UNIQUENESS

INTRODUCTION

Disclaimer: These notes are still in progress. A question of fundamental importance is the study of differential equations is the existence and uniqueness of solutions to an ordinary differential equation subject to various initial conditions. Before we begin our study, let's outline exactly what we mean in terms of this statement of "existence and uniqueness".

Let's say that we would like to study the following initial value problem:

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0. \tag{1}$$

When we discuss the **existence** of a solution, we are basically asking, *does there exist a function $y(x)$ (either in explicit or implicit form) such that $y(x)$ solves both the differential equation $y' = f(x, y)$,* ***and*** *satisfies the initial condition.*

When we discuss the **uniqueness** of a solution, we are asking whether or not there are more there is more than one solution to the differential equation that satisfies the initial conditions. Specifically, we want to know if there are two **different** functions $u_1(x)$ and $u_2(x)$ such that $y(x) = u_1(x)$ and $y(x) = u_2(x)$ both solve the ODE, and both satisfy the initial conditions.

EXAMPLE

While we could just simply state the theorem, I'd like to give you a detailed example of discussing the existence and uniqueness of solutions through the following example.

Discuss the existence and uniqueness the solution to the differential equation

$$xy' - y = 2x^3, \qquad y(x_0) = y_0$$

Since the above differential equation is linear, we can solve it using a linear integrating factor.

1. Write the ODE in standard form
$$y' - \frac{1}{x}y = 2x^2$$

2. Determine the linear integrating factor $\mu(x)$ give by $\mu = \exp\left[\int -\frac{1}{x}\,dx\right]$
$$\mu(x) = e^{-\int \frac{1}{x}\,dx} = e^{-\ln|x|} = \frac{1}{|x|}$$

   Note that we're being more careful than normal by including the absolute values with the integral. Now, there are some issues we need to be careful about at this point. The integrating function $\mu(x)$ is not differentiable at the point $x = 0$. So, for now, we will ignore the case where $x = 0$, and will focus on when $|x| > 0$. Now, with the absolute value, we have the following cases for $\mu(x)$:
$$\mu(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0, \\ \frac{-1}{x} & \text{if } x < 0, \\ \text{undefined} & \text{if } x = 0. \end{cases} \tag{2}$$

3. Multiply the ODE in standard form by $\mu(x)$ so that the LHS can be written as a product form. We will only consider the two cases where $x > 0$ and $x < 0$. We will discuss the case of $x = 0$ later.

   The case of $x > 0$:

   If $x > 0$, then $\mu(x) = \frac{1}{x}$. We can then multiply the ODE
$$\frac{1}{x}y' - \frac{1}{x^2}y = 2x$$
$$\frac{d}{dx}\left(\frac{1}{x}\cdot y\right) = 2x$$

   The case of $x < 0$:

   If $x > 0$, then $\mu(x) = \frac{1}{x}$. We can then multiply the ODE by this
$$-\frac{1}{x}y' + \frac{1}{x^2}y = -2x$$
$$-\frac{d}{dx}\left(\frac{1}{x}y\right) = -2x$$

   Upon close inspection, the two equations given above are actually the same. The one on the right is the same as the one on the left. Thus, as long as $x \neq 0$, we have the following:
$$\frac{d}{dx}\left(\frac{1}{x}\cdot y\right) = 2x.$$

   Integrating both sides with respect to x, we find
$$\frac{1}{x}y = x^2 + c.$$

   Solving for $y(x)$ we get
$$y(x) = x^3 + cx.$$

4. Solve the initial value problem. Using the initial conditions, we can now really start to discuss existence and uniqueness of solutions to this IVP in terms of the initial condition $y(x_0) = y_0$. Plugging this into our solution given above, we get
$$y_0 = x_0^3 + cx_0 \quad c = \frac{y_0 - x_0^3}{x_0}$$

   This is only valid as long as $x_0 \neq 0$. The important thing here is that the solution will exist, and we only found one solution to the initial values problem so long as $x_0 \neq 0$. Thus, the solution to the initial value problem exists and is unique.

Of course, here's the interesting question. **What happens when $x_0 = 0$?**

In order to get a better sense of what happens in this case, let's back up to our earlier solution for $y(x)$. Here we found

$$y(x) = x^3 + cx.$$

If we assume that $x_0 = 0$, then we find

$$y(0) = 0 + c \cdot 9$$

This means that for the equation to be consistent, we would need $y(0) = 0$. Thus, if we had the initial condition $y(0) = 0$, we see that there is no restriction of the value of c (try solving, you will see that any value of c will work.). Thus, we find an infinite number of solutions of the form $y(x) = x^3 + cx$ that solve $xy' - y = 2x^3$ with the initial conditions $y(0) = 0$. This means that for this set of initial conditions, we have existences of a solution, but not uniqueness.

Finally, if we have the initial conditions $y(0) = y_0$ where $y_0 \neq 0$, it would be impossible to find a solution. Thus, we do not have existence.

Therefore, we can gain some insight into how things might play out.

Theory for Linear First-Order ODEs

Now that we have trudged our way through an example the following shouldn't be too big of a leap. For linear differential equations, it turns out that we can say a lot (but not everything) about existence and uniqueness of solutions based solely on the $p(x)$ and $g(x)$ functions. This is great. It basically means that we don't have to go through all of the effort of solving the differential equation just to see if there is, or isn't a solution to the IVP!

Theorem

*Consider the following first-order, **linear**, ordinary differential equation subject to the initial conditions given:*

$$y' + p(x)y = g(x), \qquad y(x_0) = y_0,$$

*where $p(x)$ and $g(x)$ are continuous on an open interval I (i.e. $a < x < b$) that also contains x_0 (i.e. $a < x_0 < b$). Then, there exists exactly one solution $y(x) = \phi(x)$ of the differential equation $y' + p(x)y = g(x)$ **that also satisfies** the initial conditions $y(x_0) = y_0$. The solution is guaranteed to exist and be unique for all x in the interval I.*