MATH 234 - LECTURE NOTES†

FIRST ORDER EXACT DIFFERENTIAL EQUATIONS

OVERVIEW

Let's consider the following differential equation:

$$2xyy' + 2x + y^2 = 0.$$

Now, this is a pretty tough looking equation: it's nonlinear, and not separable. This means that we do not have a tool to determine the solutions to the differential equation. It would be a shame if there wasn't a method to solve for it. There is a method!

Lets look at this problem in more generality. Suppose we have a function $y(x)$ defined by the implicit equation

$$f(x, y) = c,$$

where c is a constant. Taking the x-derivative of the above equation and using the chain rule we get

$$f_x + f_y \frac{dy}{dx} = 0.$$

Thus we have an equation of the form

$$N(x, y)y' + M(x, y) = 0.$$

Thus, if there exists some implicit function $f(x, y) = c$ such that $N(x, y) = f_y$ and $M(x, y) = f_x$, the differential equation is **exact**, and the solution is given by $f(x, y) = c$.

Before we try to find $f(x, y) = c$, we should find a way to determine if a differential equation is indeed exact. Let's assume that $M(x, y) = f_x$ and $N(x, y) = f_y$. It should be true that $f_{xy} = f_{yx}$. In other words, the mixed derivatives should be equivalent. This means that $f_{xy} = (M(x, y))_y$ and $f_{yx} = (N(x, y))_x$. In other words, if we are given a first order nonlinear ODE in the form:

$$N(x, y)y' + M(x, y) = 0,$$

the equation is exact if and only if

$$(N(x, y))_x = (M(x, y))_y.$$

DEFINITION: EXACT EQUATION

An ODE of the form

$$N(x, y)y' + M(x, y) = 0,$$

is exact if and only if

$$(N(x,y))_x = (M(x,y))_y.$$

If the ODE is exact, then the solution in implicit form is $f(x,y) = c$ where $f(x,y)$ can be found by solving

$$N(x,y) = f_y \quad \text{and} \quad M(x,y) = f_x$$

Let's see how we can use this to solve the problem we started with.

Example

Consider the differential equation

$$2xyy' + 2x + y^2 = 0.$$

Solution

First we check if the equation is exact. This means that we need to check to see if $M_y = N_x$. For this problem, $M = 2x + y^2, N = 2xy$. To check for exactness, we check that

$$M_y = 2y, \text{ and } N_x = 2y$$

Since these are equal, the equation is exact. Now we can proceed to solve the differential equation. This means that we know our solution in implicit form is $f(x,y) = c$, and that

$$\begin{aligned} f_x &= 2x + y^2 \\ f_y &= 2xy \end{aligned}$$

Our goal is to solve the set of above equations for the function $f(x,y)$. We can solve these equations in the order we prefer. Let's start with the first equation $f_x = 2x + y^2$. This can be integrated with respect to x to find

$$f(x,y) = x^2 + xy^2 + h(y)$$

Note that since we are integrating with respect to x, we treat y as a constant. This means that our constant of integration is really a function of y (so it's not really a constant after all!)
We now substitute this in the second equation $f_y = 2xy$. This gives

$$2xy + h'(y) = 2xy$$

This means that $h'(y) = 0$ and $h(y) = \tilde{c}$ where $\tilde{c}$ is a real constant.
Now we can write down what we found for $f(x,y)$, to write the solution of the differential equation:

$$\boxed{f(x,y) = x^2 + 2y^2x = c,}$$

where we have absorbed $\tilde{c}$ into the original constant c.

Example

Let's do a more complicated example. Consider

$$(\sin x + x^2 e^y - 1)y' + (y\cos x + 2xe^y) = 0,$$

a nonlinear differential equation if ever there was one.

Solution

Here

$$M = y\cos x + 2xe^y, N = \sin x + x^2 e^y 1.$$

Let's check if this differential equation is exact:

$$M_y = \cos x + 2xe^y, \quad \text{and} \quad N_x = \cos x + 2xe^y.$$

These are equal, thus the equation is exact. Thus

$$f_x = y\cos x + 2xe^y$$
$$f_y = \sin x + x^2 e^y 1$$

Using the first equation $f_x = y\cos x + 2xe^y$

$$f = \int (y\cos x + 2xe^y)\ dx + h(y)$$
$$f = -y\sin x + x^2 e^y + h(y)$$

Calculating f_y, we have

$$f_y = \sin x + x^2 e^y + h'(y).$$

Plugging this in the second equation gives $h'(y) = -1$ or $h(y) = -y + \tilde{c}$. Thus, our final solution is given by the implicit equation

$$\boxed{f(x,y) = c \rightarrow -y\sin x + x^2 e^y - y = c}$$

where $\tilde{c}$ has been absorbed into the constant c.

What if the ODE is not Exact?

If an ode of the form

$$n(x,y)y' + m(x,y) = 0,$$

is not exact, we would like to be able to still try to solve it! So, we will take a cue from Leibniz and try to find a way to make the ODE exact. In other words, can we find a $\mu(x,y)$ so that

$$\mu(x,y)n(x,y)y' + \mu(x,y)m(x,y) = 0,$$

is exact. This means that

$$(\mu(x,y)n(x,y))_x = (\mu(x,y)m(x,y))_y.$$

Typically, we assume that μ is a function of x or y ONLY. In doing so, we hopefully can find a solution for the function μ by solving the equation that results from $(\mu n)_x = (\mu m)_y$.

For example, if μ is a function of x only, then the above condition simplifies to

$$(\mu n)_x = (\mu m)_y \quad \Longrightarrow \quad \mu_x n + \mu n_x = \mu m_y$$

We can write this as a first order ODE for μ as

$$\mu_x = \mu \frac{(m_y - n_x)}{n} \tag{1}$$

and if $\frac{(m_y - n_x)}{n}$ depends only on x, we can solve this ODE for the integrating factor μ via separation of variables.

What would Equation (1) be if we assumed that μ was a function of only y? *Hint: expand* $(\mu n)_x = (\mu m)_y$ *using the product rule for derivatives remembering that* μ *does not depend on* x *so that* $\mu_x = 0$.