

# MATH 234 - LECTURE NOTES<sup>†</sup>

## FIRST ORDER APPLICATIONS OF DIFFERENTIAL EQUATIONS

### SOME EXAMPLES FROM MECHANICS

In mechanics, you have seen a great many formulas relating quantities such as velocity, acceleration, position, etc. Whole collections of such results exist, and their derivation relies on distinct arguments in different settings. We'll show now that all such formulas all follow from one differential equation, namely, Newton's law. Newton's law equates force with mass times acceleration:

$$F = ma,$$

where  $F$  denotes the force acting on a particle,  $m$  is its mass, and  $a$  is its acceleration. How is this even a differential equation? Well, we know that the velocity is the derivative of the position  $s$ , and the acceleration is the derivative of the velocity  $v$ . This is how the derivatives come in. The general principle of dynamics is to specify forces. Once this is done, Newton's law gives us acceleration. Since the forces may depend on velocities and position, we typically have a second-order differential equation for the position as a function of time.

If we look at the important case of constant acceleration, we have

$$\frac{dv}{dt} = a \quad \rightarrow \quad v(t) = at + v_0$$

since we have assumed that the acceleration  $a$  is constant, and that the initial velocity at time  $t = 0$  is given by  $v(0) = v_0$ .

We have integrated once: we won't be stopped! Let's put in that the velocity is the derivative of the position. This gives

$$\frac{ds}{dt} = v(t) = at + v_0 \quad \rightarrow \quad s(t) = \frac{1}{2}at^2 + v_0t + s_0$$

where the initial position at time  $t = 0$  is given by  $s(0) = s_0$ .

This equation may look familiar. Note how systematically these results were derived using even the simplest differential equations!

An important special case of constant acceleration is that of a body falling under the influence of gravity close to the surface of the Earth. Then  $a = g$ , where we choose the position axis to point upward, hence gravity is represented by a negative acceleration. Our previous results become

$$s(t) = -\frac{g}{2}t^2 + v_0t + s_0.$$

If the body is falling from rest from height  $h$ , then  $s_0 = h$  and  $v_0 = 0$ . How long does it take the body to fall? We have

$$s(t) = \frac{g}{2}t^2 + h$$

This is zero when

$$h = \frac{g}{2}t^2 \quad \rightarrow \quad t = \sqrt{\frac{2h}{g}}.$$

The velocity at impact is

$$v(t) = -gt \quad \rightarrow \quad v = -\sqrt{2gh}.$$

Again, both of these results might look familiar.

## MIXTURE PROBLEMS

Consider the set-up shown in Fig. 1. We need to introduce some names so we are all talking about the same thing. What are we looking at? On the left is a pipe, through which fluid is flowing into the big tank. This fluid has a certain concentration of a solvent, such as salt. On the right is the outlet pipe of the tank, through which the mixture is leaving.

We are assuming that the concentration of the mixture is the same anywhere in the tank, so that the concentration of solvent in the mixture leaving the tank is the same as that of the concentration of solvent in the mixture in the tank.

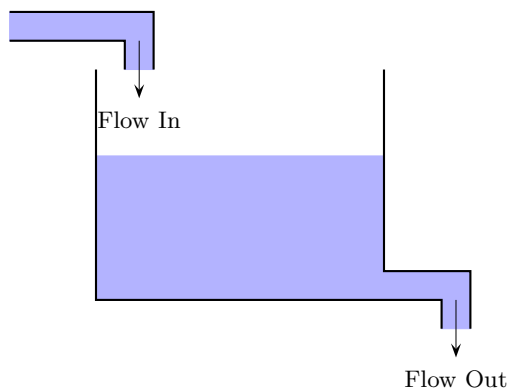


Figure 1: The set-up for the mixing problem

What names should we introduce?

- $m(t)$ : the amount of solvent at time  $t$  in the tank. This will be in kilograms (kg).
- $m_0$ : the starting amount of solvent  $m(0)$ , also in kg.
- $V(t)$ : the volume of fluid mixture in the tank at time  $t$  (liters).
- $V_0$ : the initial volume of fluid mixture  $V(0)$  (liters).
- $C_1$ : The concentration of solvent in the incoming fluid (kg/liters).

- $C_2$ : The concentration of solvent in the outgoing fluid (kg/liters).
- $q_1$ : the inflow rate (how fast is the fluid coming in; liters/sec).
- $q_2$ : the outflow rate (how fast is the fluid going out; liters/sec).

Thus, the problem to be solved is to determine  $m(t)$  for any time  $t > 0$ .

To find a differential equation that we can solve for  $m(t)$ , we use the following general rule:

**The Total Rate of Change = Rate In - Rate Out**

Since we want to find  $m(t)$ , we will use the above general rule as a guide. This means

$$\frac{dm}{dt} = \text{Rate at which } m(t) \text{ is increasing} - \text{Rate at which } m(t) \text{ is decreasing}$$

which is really just the definition of what a derivative is. So, if we can figure out suitable expressions for the increase and the decrease of  $m(t)$ , we are all set.

**- Rate of Increase:**

$$\text{Rate In} = (\text{concentration coming into the tank}) \times (\text{how fast it is entering the tank}) = C_1 q_1$$

**- Rate of Decrease:**

$$\text{Rate Out} = (\text{concentration leaving the tank}) \times (\text{how fast it is leaving the tank}) = C_2 q_2$$

Combining this information, we have that

$$\frac{dm}{dt} = C_1 q_1 - C_2 q_2.$$

Some of the things in this equation we know:  $C_1$ ,  $q_1$  and  $q_2$ . The only one we don't know is  $C_2$  (the concentration of the mixture that is flowing out). Thus, so far we have one equation, but it has two things we do not know ( $m(t)$  and  $C_2$ ). If we can determine  $C_2$  in terms of things we know, or else in terms of the other thing we do not know  $m(t)$ , we are all set! We will have one differential equation for  $m(t)$  that we can solve and have our answer! It's a cunning plan.

So, how do we determine  $C_2(t)$ ? Well,

$$C_2 = \frac{\text{amount of solvent in the tank at time } t}{\text{Volume of the Tank}} = \frac{m(t)}{V(t)}$$

But what is  $V(t)$ ? Well, to determine  $V(t)$ , we can play the same game:

$$\frac{dV}{dt} = \text{Rate In} - \text{Rate Out} = q_1 - q_2.$$

This is a very easy differential equation to solve, especially if  $q_1$  and  $q_2$  are both constant. Let's assume that both  $q_1$  and  $q_2$  are constant. Then the differential equation is separable. We can solve it (along with the initial condition  $V(0) = V_0$ ) to find

$$V(t) = (q_1 - q_2)t + V_0$$

We can already see a few interesting things from this equation:

1. If  $q_1 > q_2$ , more liquid is entering the tank than leaving it. In other words, the volume increases.
2. If  $q_1 < q_2$ , more liquid is leaving the tank than entering it. In other words, the volume decreases.
3. Lastly, if  $q_1 = q_2$  there is as much going out as coming in, and the volume in the tank stays the same.

Now we use the expression we just found in the differential equation for  $m(t)$ , to get

$$\frac{dm}{dt} = C_1 q_1 - \frac{q_2 m}{(q_1 - q_2)t + V_0}$$

This is a linear, first-order differential equation of the kind we learned how to solve by using integrating factors. In order to solve it, we first rewrite the equation in the form

$$y' + p(t)y = g(t)$$

so that we can identify  $p(t)$ , and  $g(t)$ .

$$\frac{dm}{dt} + \frac{q_2 m}{(q_1 - q_2)t + V_0} = C_1 q_1$$

By rewriting the differential equation, it's easy to identify the value of  $p(t)$  and  $g(t)$ . They are

$$p(t) = \frac{q_2}{(q_1 - q_2)t + V_0}, \text{ and } g(t) = C_1 q_1$$

Remember that the solution to the differential equation  $y' + p(t)y = g(t)$  is given by

$$y(t) = \frac{1}{\mu(t)} \left( \int \mu(t)g(t) dt + c \right), \text{ where } \mu(t) = e^{\int p(t) dt}$$

You see there are two cases:

**Case (a):**  $q_1 = q_2$  (the case of constant volume). In this case there is no  $t$ -dependence in the denominator of the function  $p(t)$ . This means that it is easy to calculate the value of the function  $\mu(t)$ . We find that

$$\mu(t) = e^{\int p(t) dt}$$

$$\mu_a(t) = e^{\frac{q_2}{V_0} t}$$

where the subscript on  $\mu_a(t)$  will be used to indicate that this is the  $\mu(t)$  value for **Case (a)** (where  $q_1 = q_2$ ).

**Case (b):**  $q_1 \neq q_2$  (the case where the volume changes). In this case, the integration is a bit more complicated. We still know that  $\mu(t) = \exp \left[ \int p(t) dt \right]$ . Calculating the integral in the exponent, we have

$$\begin{aligned} \int p(t) dt &= \int \left( \frac{q_2}{(q_1 - q_2)t + V_0} \right) dt \\ &= \frac{q_2}{q_1 - q_2} \ln((q_1 - q_2)t + V_0) \end{aligned}$$

where we have ignored the constant of integration since we are calculating the quantity  $\mu(t) = \exp \left[ \int p(t) dt \right]$  and we only need *one* such function  $\mu(t)$ .

Simplifying  $\mu(t) = \exp \left[ \int p(t) dt \right]$  we have

$$\begin{aligned} \mu(t) &= \exp \left[ \frac{q_2}{q_1 - q_2} \ln((q_1 - q_2)t + V_0) \right] \\ \mu_b(t) &= ((q_1 - q_2)t + V_0)^{\frac{q_2}{q_1 - q_2}} \end{aligned}$$

where the subscript on  $\mu_b(t)$  will be used to indicate that this is the  $\mu(t)$  value for **Case (b)** (where  $q_1 \neq q_2$ ). Note that  $\mu_b(t)$  only makes sense as long as the volume is positive (recall that  $V(t) = (q_1 - q_2)t + V_0$ ). If the volume is decreasing ( $q_2 > q_1$ ), at some point, the volume will become zero. After that, this calculation stops making sense.

Regardless of which case we are calculating, we end up with the solution

$$m(t) = \frac{1}{\mu(t)} \left( \int \mu(t)g(t) dt + c \right)$$

where  $c$  is the constant due to integration. Let's see what happens in **Case (a)** (where  $q_1 = q_2$ ).

**Case (a):** Recall from earlier we found

$$\mu_a(t) = e^{\frac{q_2}{V_0} t}.$$

Using this (in combination with the solution form for  $m(t)$  and the initial condition  $m(0) = m_0$ ), we have

$$\begin{aligned} m(t) &= \frac{1}{\mu_a(t)} \left( \int \mu_a(t)g(t) dt \right) + c \\ &= \frac{1}{e^{\frac{q_2}{V_0} t}} \left( \int C_1 q_1 e^{\frac{q_2}{V_0} t} dt + c \right) \\ &= e^{-\frac{q_2}{V_0} t} \left( q_1 C_1 \frac{V_0}{q_2} e^{\frac{q_2}{V_0} t} + c \right) \\ &= C_1 V_0 \frac{q_1}{q_2} + c e^{-\frac{q_2 t}{V_0}} \end{aligned}$$

Note that we know  $q_1 = q_2$ , we can simplify the  $\frac{q_1}{q_2} = 1$ . Also, using the initial condition that  $m(0) = m_0$ , we can solve for the constant of integration  $c$  to get the final answer

$$m_a(t) = C_1 V_0 + (m_0 - C_1 V_0) e^{-\frac{q_2 t}{V_0}},$$

where again we have used the subscript  $m_a(t)$  to represent that this is the solution for **Case (a)** (where  $q_1 = q_2$ ).

Having obtained this result, we can analyze it to obtain interesting results. Also, we should check that the outcome agrees with the intuition we have about the problem. For instance, since we keep on pouring in concentration  $C_1$ , it seems reasonable that the eventual concentration  $C_2 = m(t)/V(t)$  should approach  $C_1$ . Let's see what happens:

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{m_a(t)}{V(t)} &= \lim_{t \rightarrow \infty} \frac{C_1 V_0 + (m_0 - C_1 V_0) e^{-\frac{q_2 t}{V_0}}}{V_0} && V(t) = V_0 \text{ since } q_1 = q_2 \\ &= \lim_{t \rightarrow \infty} \left( C_1 + \frac{(m_0 - C_1 V_0)}{V_0} e^{-\frac{q_2 t}{V_0}} \right) \\ &= C_1 \end{aligned}$$

which is what we expect!

## POPULATION MODELS

Let's consider another application of first-order differential equations. This model will come from population dynamics.

### LINEAR MODEL

Suppose we are examining the growth of a small population, with plenty of resources and no predators. Denote this population by  $y(t)$ . Then, it seems reasonable that

$$\frac{dy}{dt} = ry,$$

where  $r > 0$  is the growth constant: the population increases over time.

We are stating that the population growth is proportional to the size of the population. Notice that this is a **separable ODE**. Solving this equation, as before, gives

$$y(t) = y_0 e^{rt}$$

and the population grows exponentially with time. Notice that we have used  $y_0 = y(0)$  to denote the initial population. You should be able to solve this differential equation on your own.

Now, let's consider the following example:

A population of cells grows doubles every 15 minutes. If the initial number of cells at time  $t = 0$  is 50 cells, find the number of cells after  $t$  hours.

This is an example of exponential growth, so that we can modeling it with the differential equation  $y' = ry$  with the solution  $y(t) = y_0 e^{rt}$ . Now, in order to find the population after  $t$  hours, we need to find  $y_0$  and  $r$ . From the information in the above, it's easy to determine  $y_0$ . It's just the initial population so that  $y_0 = 50$ . Now, the only thing we need to determine is the value of the growth rate  $r$ .

To find the growth rate, we will need to use the fact that the population size doubles every 15 minutes. This means that when  $t = .25$  (15 minutes = .25 hours), the population will double from 50 cells to 100 cells. In other words, we can use the fact that we know that  $y(.25) = 100$  to solve for  $r$ . Solving, we get  $r = \frac{\ln 2}{.25}$ , and so our final equation for the population of cells is given by

$$y(t) = 50e^{\frac{\ln(2)}{.25}t} = 50e^{4t \ln(2)} = 502^{4t}$$

This is reasonable, with the assumptions we have put in place. However, the above solution says that the population will experience very rapid growth. Eventually, maybe after a long time, the amount of resources might not be sufficient to support this growth. Now what happens? To include effects like this, we need a nonlinear model.

## NONLINEAR MODEL

We'll modify our model as follows:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

Using this new formulation <sup>1</sup>, we may think of  $r\left(1 - \frac{y}{K}\right)$  as new growth constant, as for the linear model. Now, the growth constant is dependent on  $y$  (so, it's not really a constant anymore): for  $y < K$ , it's positive and it looks like the population will grow. On the other hand, for  $y > K$ , the growth constant is negative and the population will decrease.

Let's solve this new, nonlinear differential equation. Note that it is a separable equation:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

$$\int \frac{dy}{\left(1 - \frac{y}{K}\right)y} = \int r dt$$

$$\int \frac{dy}{y} + \int \frac{dy}{K\left(1 - \frac{y}{K}\right)} = \int r dt \quad \text{partial fractions (see review)}$$

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + c \quad c \text{ is the constant of integration}$$

After some algebra fun (and using rules of natural logarithms), we have (check this)

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

where we have let  $y(0) = y_0$ .

Let's examine this solution to find out what happens to the population as  $t \rightarrow \infty$ : does it die out? Does it persist? Does it grow forever?

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<sup>1</sup>Generally, people refer to this nonlinear model of a population as Logistic Growth.

- If  $y_0 = 0$ , then the  $\lim_{t \rightarrow \infty} y(t) = 0$ . In other words, if there no population to start with, things are not going to be interesting.
- If  $y_0 = K$ , then the  $\lim_{t \rightarrow \infty} y(t) = K$ . Apparently  $K$  is that population level that is in perfect balance with its surroundings! There is no growth or decline of the population.
- If  $0 < y_0 < K$ , then  $\lim_{t \rightarrow \infty} y(t) = K$ . This implies that if we start with a small population (i.e., less than  $K$ ), the population grows towards the balance population.
- If  $y_0 > K$ , then  $\lim_{t \rightarrow \infty} y(t) = K$ . If we start with a population that is too large to be sustained by the available resources, the population decreases towards the balance population.

The information above is often the main information we are interested in obtaining: what will happen with the solutions if we wait long enough? And how does this depend on the initial conditions? Gee, that was quite a bit of work to get that information! Is there an easier way to get it?

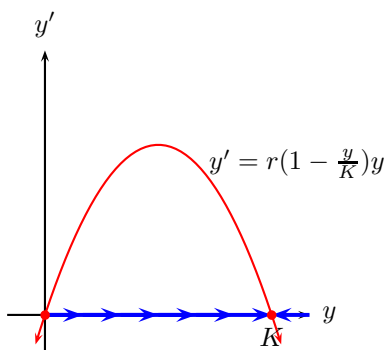
One option is to pull out the `dfield` online application to get an idea of how the solutions behave. However, it shouldn't surprise you that we can determine the same information by just looking at the differential equation itself!

Our motivation will be the following material from Calculus I:

- If the derivative of a function is positive, the function is *increasing*
- If the derivative of a function is negative, the function is *decreasing*

By using the above information, we can determine the trend of solutions without solving this equation! We will do this by drawing what we will refer to as the **Phase Line**.

The **phase-line picture** is just a plot of  $y'$  as a function of  $y$ . In other words we plot the right-hand side of the differential equation. This is done in the figure below.



This simple figure tells us a lot. It shows that whenever  $y$  is between 0 and  $K$ , the graph is positive, which means that  $y' > 0$ . This implies that  $y$  will increase. Hence, if we start somewhere on this interval, with a certain  $y$  value, we will move to  $y$ -values that are more to the right, until we reach  $y = K$ . At that point, we stop because there the graph has a zero, which implies  $y' = 0$ , thus  $y$  does not change anymore. Similarly, if  $y$  starts off to the right of  $y = K$ , the graph is negative, which means that  $y' < 0$ ,  $y$  will decrease: we will



move to the left, again until we reach  $y = K$ . We could also investigate what happens for  $y < 0$ , but such populations are not very interesting.

The two values  $y = 0$  and  $y = K$  stand out, because for these values the graph is zero, thus  $y' = 0$ , and there is no change once these values are attained. At these  $y$ -values there is no change. Such values are called **fixed points, equilibrium solutions, or equilibrium points**.

In general, an **equilibrium point** of a differential equation is a constant solution of a differential equation, such that  $y' = 0$ .

Here are some more definitions:

- An equilibrium point is called **asymptotically stable** if solutions close to it, get closer to it.
- An equilibrium point is called unstable if solutions close to it get further away.
- An equilibrium point is called semi-stable if some nearby solutions get further away, and if others get closer.

## SECOND EXAMPLE

D'Arcy Wentworth Thompson, a noted scientist of natural history, asked in his book *On Growth and Form* (1917): "But why, in the general run of shells, all the world over, in the past and in the present, one direction of twist is so overwhelmingly commoner than the other, no man knows." Most snail species are dextral (right-handed) in their shell pattern. Sinistral (left-handed) snails are exceedingly rare.

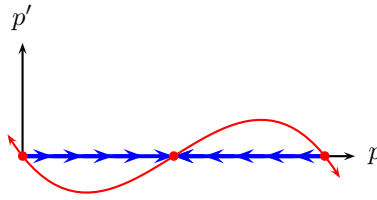
A plausible model for the appearance of such a bias in population handedness can be as follows: Let  $p(t)$  be the ratio of dextral snails in the population of snails.  $p = 1$  means that all snails are right-handed, and  $p = 0$  means that all snails are left-handed. A model equation for  $p(t)$  can be

$$p' = \alpha p(1 - p)(p - 1/2)$$

which has no left-right bias. The question that we want to pose is the following:

Suppose at  $t = 0$  (which is a very long time ago, perhaps a few hundred million years),  $p(0) = 1/2$ , that is, the dextral and sinistral snails are evenly divided. Describe what will happen a few hundred million years later. Argue that we should not expect that  $p(t) = 1/2$  as  $t \rightarrow \infty$  (i.e. equal number of dextral and sinistral snails at present time), and argue that our present state of affairs (mostly dextral snails) is an accident (i.e. we could just as well have mostly sinistral snails now).

To answer this question, it might seem like we need to solve the differential equation. However, we can use the **phase line** to avoid solving the differential equation! Let's look at the plot of  $p'$  vs.  $p$



Notice that there are exactly three equilibrium points! They are  $p^* = 0$ ,  $p^* = 1/2$ , and  $p^* = 1$ . By examining the figure, the equilibrium point  $p^* = 1/2$  is *unstable*. This means that all nearby solutions  $p(t)$  move *away* from  $p = 1/2$ . Thus, the only way to have a population that is evenly divided is to start with a population that is evenly divided. However, if there is any slight change in the system (a sinistral snail dies too early), the solution  $p(t)$  will respond by moving away from the equilibrium point. Thus, we shouldn't physically expect that the solution will remain at the point  $p(t) = 1/2$  for all time.

## HIV MODELING

*Reference: "Mathematical Analysis of HIV-1 Dynamics in Vivo" by Alan S. Perelson, and Patrick W. Nelson. SIAM Review Vol. 41 No. 1 pp 3-44*

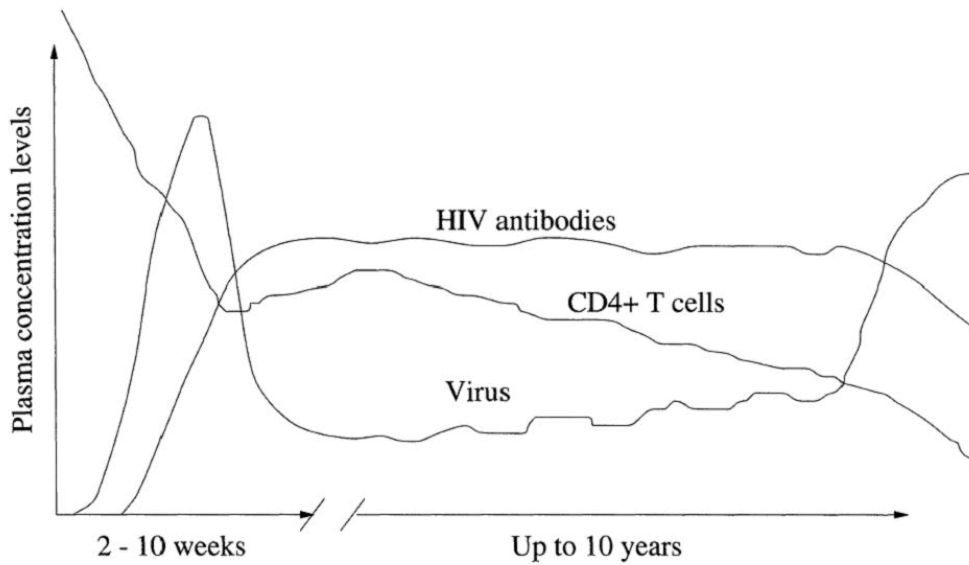
In the early days of HIV/AIDS research, researchers knew that HIV transitioned to AIDS, but had no idea at how the rate to transition occurred.

Just to bring you up to speed: HIV targets a class of lymphocytes or  $CD4^+T$  cells. These cells secrete growth/differentiation factors that are required by other cells to support the immune system. So the virus causes people to be more susceptible to pneumonia, etc.

The following is an excerpt from *"Mathematical Analysis of HIV-1 Dynamics in Vivo"* by Alan S. Perelson, and Patrick W. Nelson. *SIAM Review Vol. 41 No. 1 pp 3-44*

When the concentration of  $CD4^+T$  cells drops to about 20% of a normal concentration, the person is said to have transitioned from HIV to AIDS. The typical course of HIV infection is shown in figure (). Immediately after infection, the amount of virus detected in the blood  $V$ , rises dramatically. Along with this rise in virus, flu-like symptoms tend to appear. After a few weeks to months, the symptoms disappear, and the virus concentration falls to a lower level. An immune response to the virus occurs and antibodies against the virus can be detected in the blood. A test to detect these antibodies is used to determine if a person has been exposed to HIV. The level the virus falls to after "primary infection" has been called the set-point. The viral concentration deviates little from this set-point level for many years, however the concentration of  $CD4^+T$  cells measured in the blood slowly declines. This period in which the virus concentration stays relatively constant but in which the T cell count slowly falls is typically a period in which the infected person has no disease symptoms. The asymptomatic period can last as long as 10 years.

When this trend was discovered, people started wondering what really is going on during these asymptomatic 10 years. Was it like herpes in that it lays dormant in nerves only becoming active for some period of time?



To test this, they had to see what happened to the virus load, and the T cell count when they perturbed the system. In this case, perturb means that they needed to change the way that the host-virus was interacting during this 10 year period.

During the mid 90's, when this research was taking place, protease inhibitors were being developed and tested. Protease inhibitors work by inhibiting the activity of protease, an enzyme used by virions for the creation of new virions. After giving the drug to several patients, researchers measured the viral load, and determined the mean rate of change, and noticed that the clearance rate was proportional to the viral load. In other words, the clearance rate could be modeled by  $cV$ .

This led the researchers to a simple model

$$\frac{dV}{dt} = P(t) - cV$$

where  $V$  is the concentration of the virus,  $P(t)$  is the production on new virions, and  $c$  is the clearance rate (how fast the body gets rid of the virus).

Remember, the question we want to answer what is going on with the production of new virions during this dormant stage. So, let's consider how the system responds to the protease inhibitors. Best case scenario, the protease inhibitors cause a stop in the production on new virions. Thus,  $P(t) = 0$ . This greatly simplifies our model, and thus, we are left with

$$\frac{dV}{dt} = -cV \quad V(t) = V(t_0)e^{-c(t-t_0)}$$

The experiment was conducted by giving patients protease inhibitors, and measuring their viral load A LOT. It was determined that  $V(t_0) \approx 3.0 \times 10^5$  virions per mL. They were able to determine the half life by looking at the individual graphs (see figure (2)). On average, the half-life was  $\tau = .26$  with variability of  $\pm .08$ . Thus, using the fact that  $c = \ln(2)/\tau$ , they found  $c \approx 2.67$ .

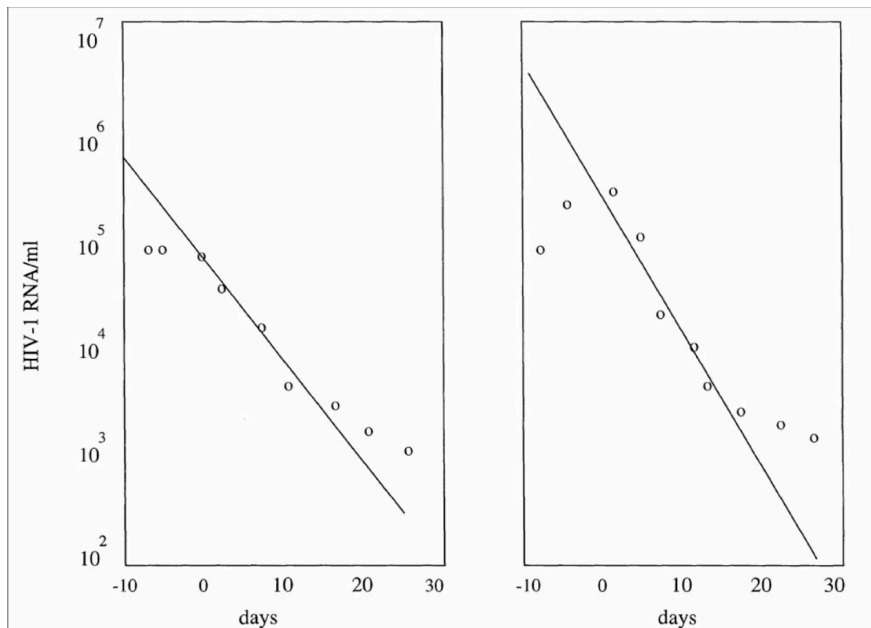


Figure 2: Plasma load before and after treatment with a protease inhibitor. Data from 2 patients (Perelson and Nelson)

So, now we have the model that once protease inhibitors are introduced into the system, the concentration of virions is modeled by

$$\frac{dV}{dt} = -2.67V$$

Now, we can try to answer our original question. How fast is the virus being produced when no treatment is available during the asymptomatic phase. To answer this question, we have to go back to the original model where  $P$  is now an unknown variable.

During the asymptomatic phase, it looks like  $V(t)$  is a constant. This indicates that  $V' = 0$ . So now, we have

$$\frac{dV}{dt} = 0 = P(t) - cV(t)$$

Since we know that right before treatment is administered  $V(t_0) = 3.0 \times 10^5$  virions per mL, we can find,  $P(t_0)$ !. Simply plug in the values that we know

$$P(t_0) = cV(t_0) = 2 \cdot 3.0 \times 10^5 = 6 \times 10^5 \text{ virions per mL per DAY!}$$

By discovering that during these early stages the virus was reproducing at a rate of nearly 1 billion per liter per day, researcher were led to determine that the best course of treatment was very aggressive treatment early!

## RADIOACTIVE DECAY

Radioactive decay is a process through which an isotope of an element transforms into another isotope of the same element. Typically, a sample of any species will be a mixture of different isotopes. By measuring how much of a certain isotope has decayed, we can determine the age of the sample, for instance. This is the basis for the method of carbon dating.

Radioactive decay is governed by a simple first-order differential equation

$$\frac{dN}{dt} = -\kappa N.$$

This equation states that the rate of change of the amount of isotope  $N$  is decaying (the minus sign on the right) proportional to the amount of the isotope: if there's a lot of the isotope, lots will decay. If there's only a little, only a small amount will decay. The constant  $\kappa$  is known as the **decay constant**. The solution of this simple differential equation is (check this, or even better: find it!)

$$N(t) = N_0 e^{-\kappa t}$$

where  $N_0 = N(0)$  is the initial amount of isotope. The **half-life time**  $\tau$  is the time it takes for half of the amount of isotope to decay. Let's figure out what it is. At the half-life time, the time is  $t = \tau$  and the amount of remaining isotope is  $N(\tau) = \frac{N_0}{2}$ . Using this information in the equation for  $N(t)$ , we have

$$\frac{N_0}{2} = N_0 e^{-\kappa \tau}.$$

Solving for the quantity  $\tau$ , we have that the half-life time is given by

$$\tau = \frac{\ln(2)}{\kappa}$$

(you should derive this answer for yourself to verify!)

## A FUN RADIOACTIVE DECAY EXAMPLE!

Carbon dating was a technique devised in the late 1940's by W.F. Libby. He went on to win the 1960 Nobel Prize in Chemistry for this information. In order to talk about carbon dating, let's consider an example of application.

In a palace in Winchester, there is an 18ft diameter round table hanging on the wall. The table was divided into 25 sections that some speculate was divided with one for a King, and 24 for Knights. Could this table be a left over from the 5th century and have belonged to King Arthur? It's know that the table predated 1484, but could it possibly be from the 5th Century? We will discuss the technique of carbon dating, and use it to determine the answer to this question.

To begin, let's talk about the carbon process. Carbon-14 is a radioactive isotope with a half-life of 5568 years. Since Carbon-14 decays so quickly, one would expect that there would be no carbon-14 remaining on the earth. However, the earth's carbon-14 supply is replenished by cosmic rays impacting nitrogen in the earth's atmosphere. The neutron from the cosmic rays interact with the existing nitrogen in the atmosphere to produce  $^{14}\text{C}$ .

The carbon-14 then spreads because of its reaction with oxygen to form  $\text{CO}_2$ . Living objects, (such as a tree shown in the figure) absorb  $\text{CO}_2$ , and which point the carbon-14 decays. While the tree is alive, the rate of absorption and rate of decay balance out. When this is true (there is no net change in carbon), we say that the concentration of carbon-14 is in *radioactive equilibrium*. To express this mathematically, let  $C(t) = [^{14}\text{C}]$ . While the tree is alive,  $\dot{C} = 0$ .

However, when the tree dies, it no longer absorbs carbon dioxide, and thus the concentration of carbon 14 is no longer in equilibrium. Since carbon-14 decays as a radioactive process, we can model it as  $\dot{C} = -\lambda C$ . Where we can determine  $\lambda$  from the half-life of carbon-14 ( $\tau = 5568$  years). Let  $t_0$  be the time that the tree died. We can then express the rate of change of the concentration of carbon-14 as

$$\frac{dC}{dt} = \begin{cases} 0 & t \leq t_0 \\ -\lambda C & t > t_0 \end{cases}$$

Returning to the possible "Round Table", in 1976, the table was taken down from the wall, and dated using several different processes, one of which was carbon dating. Scientist measured that the rate of carbon decay was  $R(t) = 6.08$  disintegrations per minute per gram in 1976. Thus in 1976, we have  $\frac{dC}{dt} = -6.08$ . But how can we use this?

Since we know that  $\dot{C} = \lambda C$ , and that the solution to this differential equation is  $C(t) = C(t_0)e^{-\lambda(t-t_0)}$ , we can plug this into the differential equation to get

$$\frac{dC}{dt} = -\lambda C(t_0)e^{-\lambda(t-t_0)} \tag{1}$$

Since we are interested in the age of the table ( $t - t_0$ ), all we need to determine is  $C(t_0)$ , and  $\lambda$ . We can easily determine  $\lambda$  to be  $\lambda = 1.245 \times 10^{-4}$  disintegrations per year (see earlier example) or in minutes,  $2.369 \times 10^{-10}$ .

Now, how do we determine  $C(t_0)$ . In this example, we use a different approach than earlier. It's assumed that the initial rate of decay of carbon in a living piece of wood is the same as the initial rate of decay of carbon in a living piece of wood currently. This assumption is based on the assumption that cosmic rays are bombarding the atmosphere at the same rate "back then" as they are now.

Using this assumption, scientist measure the initial rate of decay of a living piece of wood as 6.68 per minute per gram of the sample. Thus, we make the assumption that at  $t = t_0$  (when the tree was killed to make the table),  $R(t_0) = 6.68$ . Plugging this into equation (1), we get

$$-6.68 = -\lambda C(t_0)e^{-\lambda(t_0-t_0)} \quad C(t_0) = \frac{6.68}{\lambda} = 2.820 \times 10^{10} \text{ grams}$$

Now that we know  $C(t_0)$ , it's easy to determine the age of the table ( $t - t_0$ ). Using (1), together with the fact that the rate currently is 6.08 disintegrations per minute per gram, we get

$$-6.08 = -\lambda C(t_0)e^{-\lambda(t-t_0)} \quad t - t_0 = -\frac{1}{\lambda} \ln \left( \frac{6.08}{\lambda C(t_0)} \right) = 3.976 \times 10^8 \text{ minutes}$$

Converting this to years, scientist discovered that the table was at most 756 years old! This dates the table at being created around 1220 AD, and clearly does not date to the time of King Arthur.

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†These lecture notes are based on a combination of sources. "Some Examples from Mechanics" and "Mixture Problems" are based on the lecture notes of Dr. Bernard Deconinck at the University of Washington (used with permission). The remainder of the notes are my own adaptations of various text-book examples of mathematical models. These lecture notes have been modified to fit this class