Solutions (continued)

(6)
$$f'(t) = 3f(t) + 2g(t)$$
$$g'(t) = 2f(t) - g(t)$$

Write the above as a system of 1st order, linear, homogeneous ODEs in matrix form

$$\begin{bmatrix} f'(t) \\ g'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} \longrightarrow \vec{u}' = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \vec{u} \quad \text{where } \vec{u} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

We can solve using eigen-value, eigen-vector method.

(*) $\boxed{\text{Step 1:}}$ Assume $\vec{u} = \vec{v}e^{\lambda t}$ where $\vec{v}$ = unknown const. 2×1 vector and λ = unknown scalar

(*) $\boxed{\text{Step 2:}}$ Plug in $\vec{u} = \vec{v}e^{\lambda t}$ and rearrange.

$$\lambda \vec{v} = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \vec{v} \longrightarrow \underbrace{\begin{bmatrix} 3-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix}}_{A-\lambda I} \vec{v} = 0$$

(*) $\boxed{\text{Step 3}}$: Solve $\det(A-\lambda I) = 0$ ← you should know <u>why</u> we solve this!

$$(3-\lambda)(-1-\lambda) - 4 = 0$$

$$\lambda^2 - 2\lambda - 7 = 0 \longrightarrow \lambda^2 - 2\lambda + 1 - 8 = 0$$

$$(\lambda - 1)^2 = 8 \longrightarrow \boxed{\lambda = 1 \pm 2\sqrt{2}} \longrightarrow \underline{\underline{\text{eigenvalues}}}$$

(*) $\boxed{\text{Step 4:}}$ For each eigenvalue, calculate the corresponding eigenvector

$\boxed{\text{For } \lambda_1 = 1 + 2\sqrt{2}}$

$$(A - \lambda I)\vec{v} = \vec{0} \implies \begin{bmatrix} 2-2\sqrt{2} & 2 \\ 2 & -2-2\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

looking at the bottom eqn:

$$2v_1 - 2(1+\sqrt{2})v_2 = 0$$

$$v_1 = (1+\sqrt{2})v_2 \implies \boxed{\vec{v}^{(1)} = \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}}$$

corresponding eigenvector!

$\boxed{\text{For } \lambda = 1 - \sqrt{2}}$

Similar calculations show $\boxed{\vec{v}^{(2)} = \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}}$

(*) $\boxed{\text{Step 5:}}$ Write the general solution

$$\vec{u} = c_1 \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} e^{(1+2\sqrt{2})t} + c_2 \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} e^{(1-2\sqrt{2})t}$$

Note: Since the problem was stated in terms of f's and g's, we should state the answer for f(t) and g(t).

$$\vec{u} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} \rightarrow \boxed{\begin{aligned} f(t) &= c_1(1+\sqrt{2})e^{(1+2\sqrt{2})t} + c_2(1-\sqrt{2})e^{(1-2\sqrt{2})t} \\ g(t) &= c_1 e^{(1+2\sqrt{2})t} + c_2 e^{(1-2\sqrt{2})t} \end{aligned}}$$

final answer

(7) $\left[\sin(x) + x^2 e^y - 1\right] y' + y\cos(x) + 2xe^y = 0$

First order, nonlinear, non-separable $\rightarrow$ Is it exact?

Given $M(x,y) + N(x,y)\,y' = 0$

If: $M_y = N_x$, then the solution can be written in implicit form as

$$f(x,y) = c$$

where $f_x = M(x,y)$ and $f_y = N(x,y)$

For this problem, $M(x,y) = y\cos(x) + 2xe^y$ and $N(x,y) = \sin(x) + x^2 e^y - 1$

$$M_y = \cos(x) + 2xe^y \qquad N_x = \cos(x) + 2xe^y + 0$$

Since $M_y = N_x$, the ODE is exact.

$$f_x = M(x,y) \rightarrow f_x = y\cos(x) + 2xe^y \quad \text{(A)}$$

$$f_y = N(x,y) \rightarrow f_y = \sin(x) + x^2e^y - 1 \quad \text{(B)}$$

Integrating (A) w.r.t x, we find

$$\int f_x \, dx = \int (y\cos(x) + 2xe^y)\, dx$$

$$f = y\sin(x) + x^2e^y + h(y) \quad \text{(C)}$$

remember, we can have an arbitrary function of y here.

To find $h(y)$, we can either integrate (B) w.r.t y and compare, or take a y-partial derivative of (C) and compare with (B).

I tend to do the latter.

$$\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\left(y\sin(x) + x^2e^y + h(y)\right)$$

$$f_y = \underbrace{\sin(x) + x^2e^y + h'(y)}_{} \rightarrow \text{this should equal (B)}$$

$$\cancel{\sin(x)} + \cancel{x^2e^y} + h'(y) = \cancel{\sin(x)} + \cancel{x^2e^y} - 1 \quad \text{(B)}$$

$$h'(y) = -1 \rightarrow h(y) = -y + \tilde{c}$$

Therefore, $f(x,y) = y\cdot\sin(x) + x^2e^y - y + \tilde{c}$.

The implicit solution is given by $f(x,y) = c$

$$\boxed{y\sin(x) + x^2e^y - y = c}$$