

①

$$\textcircled{1} \quad y' + y = 0$$

$$\frac{dy}{dx} + y = 0 \rightarrow \frac{dy}{dx} = -y$$

separate and integrate

$$\int \frac{dy}{y} = \int -dx$$

$$\ln|y| = -x + \tilde{c} \rightarrow |y| = e^{-x+\tilde{c}}$$

$$\boxed{y = ce^{-x}}$$

$$\textcircled{2} \quad y'' + y = 0$$

guess $y = e^{\lambda x}$ → plug in

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \rightarrow \text{factor out } e^{\lambda x}$$

$$(\lambda^2 + 1) e^{\lambda x} = 0 \rightarrow e^{\lambda x} \neq 0$$

$$\lambda^2 + 1 = 0 \rightarrow \lambda = \pm i \rightarrow \text{complex roots!}$$

Note: If your roots take the form $\lambda = \alpha \pm i\beta$ then the real fundamental sols take the form

$$y_1 = e^{\alpha x} \cos(\beta x), \quad y_2 = e^{\alpha x} \sin(\beta x)$$

Since $\lambda = \pm i \rightarrow \alpha = 0, \beta = 1 \rightarrow y_1 = \cos(x), y_2 = \sin(x)$

general sol:

$$\boxed{y(x) = c_1 \cos(x) + c_2 \sin(x)}$$

(3)

$$y'' + y = \cos(x) *$$

Note: this problem is non-homogeneous. Thus the solution will consist of a homogeneous solution and the particular solution.

$$y = y_h + y_p$$

From (2), we know $y_h = c_1 \cos(x) + c_2 \sin(x)$. Now we only need to find the particular solution. There are several methods, but undetermined coefficients might be fastest.

Since $g(x) = \cos(x)$, we guess $y_p = A \cos(x) + B \sin(x)$. However, this is problematic! y_p is our homog. sol!

Since our guess contains the homog. sol, we must multiply y_p by x .

Guess $y_p = Ax \cos(x) + Bx \sin(x)$ → plug this into *

↳ this is the form of the particular sol.

Note: we need to calculate y_p'' in order to plug into *

$$y'_p = A \cos(x) - Ax \sin(x) + B \sin(x) + Bx \cos(x)$$

$$y''_p = -A \sin(x) - A \sin(x) - Ax \cos(x) + B \cos(x) + Bx \cos(x) - Bx \sin(x)$$

$$(-2A \sin(x) - Ax \cos(x) + 2B \cos(x) - Bx \sin(x)) + (Ax \cos(x) + Bx \sin(x)) = \cos(x)$$

Equating like terms:

$$x \cdot \sin(x): -B + B = 0 \checkmark$$

$$x \cdot \cos(x): -A + A = 0 \checkmark$$

$$\sin(x): -2A = 0 \rightarrow A = 0$$

$$\cos(x): 2B = 1 \rightarrow B = \frac{1}{2}$$

Thus $y_p = \frac{1}{2}x \sin(x)$

$$y(x) = c_1 \cos(x) + c_2 \sin(x) + \frac{1}{2}x \sin(x)$$

$$(4) \quad y' + 2xy = \cos(x) \quad (*)$$

This eqn is linear and first order, so we can use an integrating function to solve. \rightarrow Note: (*) is already in std. form.

$$\text{Let } \mu(x) = e^{\int 2x dx} \rightarrow \mu(x) = e^{x^2}$$

If we multiply (*) by $\mu(x)$, we get

$$e^{x^2} y' + e^{x^2} \cdot 2xy = e^{x^2} \cos(x)$$

note: this is a product rule! $\frac{d}{dx} [e^{x^2} \cdot y(x)] = e^{x^2} y' + e^{x^2} 2xy$

$\Rightarrow \frac{d}{dx} [e^{x^2} y] = e^{x^2} \cos(x) \rightarrow \text{integrate both sides w.r.t. } x.$

$$e^{x^2} \cdot y(x) = \int e^{x^2} \cos(x) dx + C \quad (*)$$

Warning: don't forget this "+C".

Solving for $y(x)$ to get the explicit solution

$$y(x) = e^{-x^2} \cdot \int e^{x^2} \cos(x) dx + C e^{-x^2}$$

(*) Note: this cannot be evaluated in closed form. Therefore, the answer can be expressed in terms of the integral.

⑤

$$(x-2)y'' + xy' + y = 0$$

$$y(0) = 1, \quad y'(0) = 2$$

Non-const coefficient second order \Rightarrow Series

centered at $x_0 = 0$

$$\text{Let } y = \sum_{n=0}^{\infty} a_n x^n \rightarrow y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \rightarrow y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Plugging the above into the ODE, we get

$$x \cdot \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - 2 \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + x \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$\leftarrow \sum_{n=2}^{\infty} a_n n(n-1) x^{n-1} - \sum_{n=2}^{\infty} 2a_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

 ↓1 ↑1 ↓2 ↑2 ✓ ✓ ✓

shifting the indices so that all sums contain term like x^n

$$\sum_{n=1}^{\infty} a_{n+1} (n+1)(n) x^n - \sum_{n=0}^{\infty} 2a_{n+2} (n+2)(n+1) x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

now, we must match the starting index. In this case, we would need all series to start at $\underline{n=1}$

$$-2a_2(2)(1)x^0 + a_0x^0 + \sum_{n=1}^{\infty} x^n \left[a_{n+1}(n)(n+1) - 2a_{n+2}(n+1)(n+2) + a_n n + a_n \right] = 0$$

Now, we can equate like powers of x :

$$x^0 :$$

$$-2a_2 + a_0 \Rightarrow a_2 = \frac{1}{4}a_0$$

solve for a_2

$$x^n : \quad n \geq 1$$

$$a_{n+1}(n+1)(n) - 2a_{n+2}(n+1)(n+2) + a_n(n+1) = 0$$

solve for a_{n+2}

$$\hookrightarrow a_{n+2} = \frac{(n+1)[a_n + n a_{n+1}]}{2 \cdot (n+1)(n+2)}$$

So, we have the following:

$$a_2 = \frac{1}{4}a_0, \quad a_{n+2} = \frac{a_n + n a_{n+1}}{2 \cdot (n+2)}, \quad n \geq 1$$

(*) Note: $a_{n+2} = \frac{a_n + n a_{n+1}}{2(n+2)}$, for $n \geq 1$ is the recursion relationship.

Let's write the first 5 terms in the solution:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

using the above relationships and the initial conditions,

$$\begin{aligned} a_0 &= 1 & a_1 &= 2 & a_2 &= \frac{1}{4}a_0 = \frac{1}{4} & a_3 &= \frac{a_1 + a_2}{2 \cdot 3} = \frac{2 + \frac{1}{4}}{6} = \frac{9}{24} \\ &&&&&&= \frac{3}{8} \end{aligned}$$

$$a_4 = \frac{a_2 + 2a_3}{2(4)} = \frac{\frac{1}{4} + \frac{3}{4}}{2 \cdot 4} = \frac{1}{8}$$

$$y(x) = 1 + 2x + \frac{1}{4}x^2 + \frac{3}{8}x^3 + \frac{1}{8}x^4 + \dots$$