MATH 2340 - LECTURE NOTES[†]

REVIEW OF INTEGRATION TECHNIQUES

OVERVIEW

This handout contains a review of some of the major techniques of integration, including

- Substitution methods
- Integration by Parts
- Integrating Rational Functions

and a table of common integrals. It is based on the Oregon State University's Math Department "Review of Integration Techniques" web-page [1]. A more thorough and complete treatment of these methods can be found in any calculus book.

SUBSTITUTION METHODS

In some cases, an integral can be altered into a manageable form by just changing variables. If the integrand can be written in the form $u \cdot du$ then we can easily calculate the integral. For example, consider the following integral:

$$\int (2x+1)\cdot(x^2+x)^2 dx$$

Let $u = x^2 + x$. If we calculate the quantity du we have

$$du = (2x+1)\ dx$$

Using this information, we can substitute u and du into the integral to rewrite it as

$$\int u^2\ du$$

which is an integral we can easily calculate. Thus, we have

$$\int u^2\ du = \frac{1}{3}u^3 + c.$$

However, this is not our final answer. We must give our final answer in terms of x. Using $u = x^2 + x$, we have

$$\boxed{\int (2x+1)\cdot(x^2+x)^2 dx = \frac{1}{3}(x^2+x)^3 + c}$$

[1] http://www.math.oregonstate.edu/home/programs/undergrad/CalculusQuestStudyGuides/integration/integration.html

Integration by Parts

The key idea behind integration by parts is the use of the product rule. Let $u(x)$ and $v(x)$ be two differentiable functions. If we take the derivative of $(u(x) \cdot v(x))$, we have

$$\frac{d}{dx}(u \cdot v) = u'v + uv'$$

If we take the integral of both sides of the above equation, we have

$$uv = \int u'v dx + \int uv' dx$$

Rearranging the terms, we end up with the following rule:

$$\int uv' dx = uv - \int u'v dx$$

Integration by parts is useful in "eliminating" a part of the integral that makes the integral difficult to do. The annoying part of the integral is often chosen to be $u(x)$. Alternatively, we can choose v' to be the most complicated part of the integral that we can integrate. Consider the following example:

$$\int x^3 e^{x^2}\ dx$$

For this problem, let $v' = xe^{x^2}$ since this appears to be the most complicated function that we can integrate. This means that the remainder of the integral x^2 will be the other function $u(x)$. To use the rule given above, we also need to identify u' and v. This is easy to determine, and we now have the following quantities:

$$u(x) = x^2 \qquad v'(x) = xe^{x^2}$$

$$u'(x) = 2x \qquad v(x) = \frac{1}{2}e^{x^2}*$$

*Note that we have ignored the constant of integration here. This is because we will include it separately below. You can justify for yourself that you will get the same answer. Now we are ready to use the rule

$$\int uv' dx = uv - \int u'v dx.$$

This means that we have

$$\begin{aligned}
\int x^3 e^{x^2}\ dx &= x^2 \cdot \frac{1}{2}e^{x^2} - \int 2x \cdot \frac{1}{2}e^{x^2}\ dx \\
&= \frac{x^2}{2}e^{x^2} - \int \cdot xe^{x^2}\ dx \\
&= \frac{x^2}{2}e^{x^2} - \frac{1}{2}e^{x^2} + c
\end{aligned}$$

Simplifying, we get the final answer

$$\boxed{\int x^3 e^{x^2}\ dx = \frac{1}{2}e^{x^2}(x^2 - 1) + c}$$

Integrating Rational Functions

A rational function is a function that can be expressed as the ratio of two polynomials. Consider integrating the rational function

$$\frac{3x+2}{2x^2+x-3} = \frac{3x+2}{(2x+3)(x-1)}$$

To integrate such a function we use the method of partial fractions[2] to split the fraction into easily integrable pieces:

$$\frac{3x+2}{2x^2+x-3} = \frac{1}{(2x+3)} + \frac{1}{x-1}$$

Now the integral is easy:

$$\begin{aligned} \int \frac{3x+2}{2x^2+x-3}\,dx &= \int \frac{1}{(2x+3)}\,dx + \int \frac{1}{x-1}\,dx \\ &= \frac{1}{2}\ln(2x+3) + \ln(x-1) + c \end{aligned}$$

Thus,

$$\boxed{\int \frac{3x+2}{2x^2+x-3}\,dx = \frac{1}{2}\ln(2x+3) + \ln(x-1) + c}$$

This method is designed for fractions with a polynomial of lesser degree in the numerator than in the denominator. If this is not the case, long (or synthetic) division must be carried out first and then the method of partial fractions can be used on the remainder term (if necessary).

[2]Refer to a calculus book or the following website for review: http://sosmath.com/algebra/pfrac/pfrac.html

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REVIEW OF INTEGRATION TECHNIQUES

BASIC FORMS

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c \tag{1}$$

$$\int \frac{1}{x} dx = \ln x + c \tag{2}$$

$$\int u dv = uv - \int v du \tag{3}$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax+b| + c \tag{4}$$

INTEGRALS OF RATIONAL FUNCTIONS

$$\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a} + c \tag{5}$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1} + c, n \neq -1 \tag{6}$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} + c \tag{7}$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c \tag{8}$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c \tag{9}$$

$$\int \frac{x}{a^2+x^2} dx = \frac{1}{2} \ln|a^2+x^2| + c \tag{10}$$

$$\int \frac{x^2}{a^2+x^2} dx = x - a \tan^{-1} \frac{x}{a} + c \tag{11}$$

$$\int \frac{x^3}{a^2+x^2} dx = \frac{1}{2}x^2 - \frac{1}{2}a^2 \ln|a^2+x^2| + c \tag{12}$$

$$\int \frac{1}{ax^2+bx+c} dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} + C \tag{13}$$

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln \frac{a+x}{b+x}, \ a \neq b \tag{14}$$

$$\int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln|a+x| + C \tag{15}$$

$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} + C \tag{16}$$

Integrals with Roots

$$\int \sqrt{x-a}\,dx = \frac{2}{3}(x-a)^{3/2} + C \tag{17}$$

$$\int \frac{1}{\sqrt{x\pm a}}dx = 2\sqrt{x\pm a} + C \tag{18}$$

$$\int \frac{1}{\sqrt{a-x}}dx = -2\sqrt{a-x} + C \tag{19}$$

$$\int x\sqrt{x-a}\,dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2} + C \tag{20}$$

$$\int \sqrt{ax+b}\,dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b} + C \tag{21}$$

$$\int (ax+b)^{3/2}dx = \frac{2}{5a}(ax+b)^{5/2} + C \tag{22}$$

$$\int \frac{x}{\sqrt{x\pm a}}dx = \frac{2}{3}(x\mp 2a)\sqrt{x\pm a} + C \tag{23}$$

$$\int \sqrt{\frac{x}{a-x}}dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a} + C \tag{24}$$

$$\int \sqrt{\frac{x}{a+x}}dx = \sqrt{x(a+x)} - a\ln\left[\sqrt{x} + \sqrt{x+a}\right] + C \tag{25}$$

$$\int x\sqrt{ax+b}\,dx = \frac{2}{15a^2}(-2b^2 + abx + 3a^2x^2)\sqrt{ax+b} + C \tag{26}$$

$$\int \sqrt{x(ax+b)}\,dx = \frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)} - b^2\ln\left|a\sqrt{x} + \sqrt{a(ax+b)}\right|\right] + C \tag{27}$$

$$\int \sqrt{x^3(ax+b)}\,dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right]\sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}}\ln\left|a\sqrt{x} + \sqrt{a(ax+b)}\right| + C \tag{28}$$

$$\int \sqrt{x^2\pm a^2}\,dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \pm \frac{1}{2}a^2\ln\left|x + \sqrt{x^2\pm a^2}\right| + C \tag{29}$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} + C \tag{30}$$

$$\int x\sqrt{x^2\pm a^2}\,dx = \frac{1}{3}\left(x^2\pm a^2\right)^{3/2} + C \tag{31}$$

$$\int \frac{1}{\sqrt{x^2\pm a^2}}dx = \ln\left|x + \sqrt{x^2\pm a^2}\right| + C \tag{32}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C \tag{33}$$

$$\int \frac{x}{\sqrt{x^2\pm a^2}}dx = \sqrt{x^2\pm a^2} + C \tag{34}$$

$$\int \frac{x}{\sqrt{a^2-x^2}}dx = -\sqrt{a^2-x^2} + C \tag{35}$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{2} x \sqrt{x^2 \pm a^2} \mp \frac{1}{2} a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C \tag{36}$$

$$\int \sqrt{ax^2 + bx + c} dx = \frac{b + 2ax}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx^+ c)} \right| + C \tag{37}$$

$$\int x \sqrt{ax^2 + bx + c} = \frac{1}{48a^{5/2}} \Big(2\sqrt{a}\sqrt{ax^2 + bx + c} - \left(3b^2 + 2abx + 8a(c + ax^2)\right) +3(b^3 - 4abc) \ln \left| b + 2ax + 2\sqrt{a}\sqrt{ax^2 + bx + x} \right| \Big) \tag{38}$$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| + C \tag{39}$$

$$\int \frac{x}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{a} \sqrt{ax^2 + bx + c} + \frac{b}{2a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| + C \tag{40}$$

Integrals with Logarithms

$$\int \ln ax dx = x \ln ax - x + C \tag{41}$$

$$\int \frac{\ln ax}{x} dx = \frac{1}{2} (\ln ax)^2 + C \tag{42}$$

$$\int \ln(ax+b) dx = \left(x + \frac{b}{a} \right) \ln(ax+b) - x + C, a \neq 0 \tag{43}$$

$$\int \ln \left(a^2x^2 \pm b^2\right) dx = x \ln \left(a^2x^2 \pm b^2\right) + \frac{2b}{a} \tan^{-1} \frac{ax}{b} - 2x + C \tag{44}$$

$$\int \ln \left(a^2 - b^2x^2\right) dx = x \ln \left(ar - b^2x^2\right) + \frac{2a}{b} \tan^{-1} \frac{bx}{a} - 2x + C \tag{45}$$

$$\int \ln \left(ax^2 + bx + c\right) dx = \frac{1}{a} \sqrt{4ac - b^2} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}} - 2x + \left(\frac{b}{2a} + x \right) \ln \left(ax^2 + bx + c\right) + C \tag{46}$$

$$\int x \ln(ax + b) dx = \frac{bx}{2a} - \frac{1}{4} x^2 + \frac{1}{2} \left(x^2 - \frac{b^2}{a^2} \right) \ln(ax + b) + C \tag{47}$$

$$\int x \ln \left(a^2 - b^2x^2\right) dx = -\frac{1}{2} x^2 + \frac{1}{2} \left(x^2 - \frac{a^2}{b^2} \right) \ln \left(a^2 - b^2x^2\right) + C \tag{48}$$

Integrals with Exponentials

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C \tag{49}$$

$$\int \sqrt{x}e^{ax}dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}}\text{erf}\left(i\sqrt{ax}\right) + C, \quad \text{where } \text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dtet \tag{50}$$

$$\int xe^x dx = (x-1)e^x + C \tag{51}$$

$$\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} + C \tag{52}$$

$$\int x^2 e^x dx = \left(x^2 - 2x + 2\right)e^x + C \tag{53}$$

$$\int x^2 e^{ax}dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax} + C \tag{54}$$

$$\int x^3 e^x dx = \left(x^3 - 3x^2 + 6x - 6\right)e^x + C \tag{55}$$

$$\int x^n e^{ax}\,\mathrm{d}x = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,\mathrm{d}x \tag{56}$$

$$\int x^n e^{ax}\,\mathrm{d}x = \frac{(-1)^n}{a^{n+1}}\Gamma[1+n, -ax] + C, \quad \text{where } \Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,\mathrm{d}t \tag{57}$$

$$\int e^{ax^2}\,\mathrm{d}x = -\frac{i\sqrt{\pi}}{2\sqrt{a}}\text{erf}\left(ix\sqrt{a}\right) + C \tag{58}$$

$$\int e^{-ax^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{a}}\text{erf}\left(x\sqrt{a}\right) + C \tag{59}$$

Integrals with Trigonometric Functions

$$\int \sin ax dx = -\frac{1}{a}\cos ax + C \tag{60}$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C \tag{61}$$

$$\int \sin^n ax dx = -\frac{1}{a}\cos ax \;\; {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right] + C \tag{62}$$

$$\int \sin^3 ax dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a} + C \tag{63}$$

$$\int \cos ax dx = \frac{1}{a}\sin ax + C \tag{64}$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C \tag{65}$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)}\cos^{1+p} ax \times {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right] + C \tag{66}$$

$$\int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a} + C \tag{67}$$

$$\int \cos ax \sin bx dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)} + C, a \neq b \tag{68}$$

$$\int \sin^2 ax \cos bx dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} + C \tag{69}$$

$$\int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x + C \tag{70}$$

$$\int \cos^2 ax \sin bx dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)} + C \tag{71}$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a}\cos^3 ax + C \tag{72}$$

$$\int \sin^2 ax \cos^2 bx dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)} + C \tag{73}$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a} + C \tag{74}$$

$$\int \tan ax dx = -\frac{1}{a}\ln\cos ax + C \tag{75}$$

$$\int \tan^2 ax dx = -x + \frac{1}{a}\tan ax + C \tag{76}$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right) + C \tag{77}$$

$$\int \tan^3 ax dx = \frac{1}{a}\ln\cos ax + \frac{1}{2a}\sec^2 ax + C \tag{78}$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C = 2\tanh^{-1}\left(\tan\frac{x}{2}\right) + C \tag{79}$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax + C \tag{80}$$

$$\int \sec^3 x \,\mathrm{dx} = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C \tag{81}$$

$$\int \sec x \tan x dx = \sec x + C \tag{82}$$

$$\int \sec^2 x \tan x dx = \frac{1}{2}\sec^2 x + C \tag{83}$$

$$\int \sec^n x \tan x dx = \frac{1}{n}\sec^n x + C, n \neq 0 \tag{84}$$

$$\int \csc x dx = \ln\left|\tan\frac{x}{2}\right| + C = \ln|\csc x - \cot x| + C \tag{85}$$

$$\int \csc^2 ax dx = -\frac{1}{a}\cot ax + C \tag{86}$$

$$\int \csc^3 x dx = -\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x| + C \tag{87}$$

$$\int \csc^n x \cot x dx = -\frac{1}{n}\csc^n x + C, n \neq 0 \tag{88}$$

$$\int \sec x \csc x dx = \ln|\tan x| + C \tag{89}$$

Products of Trigonometric Functions and Monomials

$$\int x \cos x dx = \cos x + x \sin x + C \tag{90}$$

$$\int x \cos ax dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C \tag{91}$$

$$\int x^2 \cos x dx = 2x \cos x + \left(x^2 - 2\right) \sin x + C \qquad (92)$$

$$\int x^2 \cos ax dx = \frac{2x \cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3} \sin ax + C \qquad (93)$$

$$\int x^n cos x dx = -\frac{1}{2}(i)^{n+1} \left[\Gamma(n+1, -ix) + (-1)^n \Gamma(n+1, ix)\right] + C \qquad (94)$$

$$\int x^n cos ax dx = \frac{1}{2}(ia)^{1-n} \left[(-1)^n \Gamma(n+1, -iax) - \Gamma(n+1, ixa)\right] + C \qquad (95)$$

$$\int x \sin x dx = -x \cos x + \sin x + C \qquad (96)$$

$$\int x \sin ax dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2} + C \qquad (97)$$

$$\int x^2 \sin x dx = \left(2 - x^2\right) \cos x + 2x \sin x + C \qquad (98)$$

$$\int x^2 \sin ax dx = \frac{2 - a^2x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2} + C \qquad (99)$$

$$\int x^n \sin x dx = -\frac{1}{2}(i)^n \left[\Gamma(n+1, -ix) - (-1)^n \Gamma(n+1, -ix)\right] + C \qquad (100)$$

Products of Trigonometric Functions and Exponentials

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C \qquad (101)$$

$$\int e^{bx} \sin ax dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax) + C \qquad (102)$$

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) + C \qquad (103)$$

$$\int e^{bx} \cos ax dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin ax + b \cos ax) + C \qquad (104)$$

$$\int x e^x \sin x dx = \frac{1}{2} e^x (\cos x - x \cos x + x \sin x) + C \qquad (105)$$

$$\int x e^x \cos x dx = \frac{1}{2} e^x (x \cos x - \sin x + x \sin x) + C \qquad (106)$$

Integrals of Hyperbolic Functions

$$\int \cosh ax dx = \frac{1}{a}\sinh ax + C \tag{107}$$

$$\int e^{ax}\cosh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2-b^2}[a\cosh bx - b\sinh bx] + C & a \neq b \\ \dfrac{e^{2ax}}{4a} + \dfrac{x}{2} + C & a = b \end{cases} \tag{108}$$

$$\int \sinh ax dx = \frac{1}{a}\cosh ax + C \tag{109}$$

$$\int e^{ax}\sinh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2-b^2}[-b\cosh bx + a\sinh bx] + C & a \neq b \\ \dfrac{e^{2ax}}{4a} - \dfrac{x}{2} + C & a = b \end{cases} \tag{110}$$

$$\int e^{ax}\tanh bx dx = \begin{cases} \dfrac{e^{(a+2b)x}}{(a+2b)}\,{}_2F_1\left[1+\dfrac{a}{2b}, 1, 2+\dfrac{a}{2b}, -e^{2bx}\right] \\ \qquad -\dfrac{1}{a}e^{ax}\,{}_2F_1\left[\dfrac{a}{2b}, 1, 1E, -e^{2bx}\right] + C & a \neq b \\ \dfrac{e^{ax} - 2\tan^{-1}[e^{ax}]}{a} + C & a = b \end{cases} \tag{111}$$

$$\int \tanh bx dx = \frac{1}{a}\ln\cosh ax + C \tag{112}$$

$$\int \cos ax \cosh bx dx = \frac{1}{a^2+b^2}[a\sin ax\cosh bx + b\cos ax\sinh bx] + C \tag{113}$$

$$\int \cos ax \sinh bx dx = \frac{1}{a^2+b^2}[b\cos ax\cosh bx + a\sin ax\sinh bx] + C \tag{114}$$

$$\int \sin ax \cosh bx dx = \frac{1}{a^2+b^2}[-a\cos ax\cosh bx + b\sin ax\sinh bx] + C \tag{115}$$

$$\int \sin ax \sinh bx dx = \frac{1}{a^2+b^2}[b\cosh bx\sin ax - a\cos ax\sinh bx] + C \tag{116}$$

$$\int \sinh ax \cosh ax dx = \frac{1}{4a}[-2ax + \sinh 2ax] + C \tag{117}$$

$$\int \sinh ax \cosh bx dx = \frac{1}{b^2-a^2}[b\cosh bx\sinh ax - a\cosh ax\sinh bx] + C \tag{118}$$