

MATH 2340 - LECTURE NOTES[†]

REVIEW OF INTEGRATION TECHNIQUES

BASIC FORMS

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c \quad (1)$$

$$\int \frac{1}{x} dx = \ln|x| + c \quad (2)$$

$$\int u dv = uv - \int v du \quad (3)$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax+b| + c \quad (4)$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c \quad (8)$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c \quad (9)$$

$$\int \frac{x}{a^2+x^2} dx = \frac{1}{2} \ln|a^2+x^2| + c \quad (10)$$

$$\int \frac{x^2}{a^2+x^2} dx = x - a \tan^{-1} \frac{x}{a} + c \quad (11)$$

$$\int \frac{x^3}{a^2+x^2} dx = \frac{1}{2} x^2 - \frac{1}{2} a^2 \ln|a^2+x^2| + c \quad (12)$$

INTEGRALS OF RATIONAL FUNCTIONS

$$\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a} + c \quad (5)$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1} + c, n \neq -1 \quad (6)$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} + c \quad (7)$$

$$\int \frac{1}{ax^2+bx+c} dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} + C \quad (13)$$

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln \frac{a+x}{b+x}, a \neq b \quad (14)$$

$$\int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln|a+x| + C \quad (15)$$

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$$\int \frac{x}{ax^2 + bx + c} dx = \frac{1}{2a} \ln |ax^2 + bx + c| - \frac{b}{a\sqrt{4ac - b^2}} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}} + C \quad (16)$$

$$\int x\sqrt{ax + b} dx = \frac{2}{15a^2} (-2b^2 + abx + 3a^2x^2)\sqrt{ax + b} + C \quad (26)$$

INTEGRALS WITH ROOTS

$$\int \sqrt{x - a} dx = \frac{2}{3}(x - a)^{3/2} + C \quad (17)$$

$$\int \frac{1}{\sqrt{x \pm a}} dx = 2\sqrt{x \pm a} + C \quad (18)$$

$$\int \frac{1}{\sqrt{a - x}} dx = -2\sqrt{a - x} + C \quad (19)$$

$$\int x\sqrt{x - a} dx = \frac{2}{3}a(x - a)^{3/2} + \frac{2}{5}(x - a)^{5/2} + C \quad (20)$$

$$\int \sqrt{ax + b} dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax + b} + C \quad (21)$$

$$\int (ax + b)^{3/2} dx = \frac{2}{5a}(ax + b)^{5/2} + C \quad (22)$$

$$\int \frac{x}{\sqrt{x \pm a}} dx = \frac{2}{3}(x \mp 2a)\sqrt{x \pm a} + C \quad (23)$$

$$\int \sqrt{\frac{x}{a - x}} dx = -\sqrt{x(a - x)} - a \tan^{-1} \frac{\sqrt{x(a - x)}}{x - a} + C \quad (24)$$

$$\int \sqrt{\frac{x}{a + x}} dx = \sqrt{x(a + x)} - a \ln [\sqrt{x} + \sqrt{x + a}] + C \quad (25)$$

$$\int \sqrt{x(ax + b)} dx = \frac{1}{4a^{3/2}} \left[(2ax + b)\sqrt{ax(ax + b)} - b^2 \ln \left| a\sqrt{x} + \sqrt{a(ax + b)} \right| \right] + C \quad (27)$$

$$\int \sqrt{x^3(ax + b)} dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3} \right] \sqrt{x^3(ax + b)} + \frac{b^3}{8a^{5/2}} \ln \left| a\sqrt{x} + \sqrt{a(ax + b)} \right| + C \quad (28)$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C \quad (29)$$

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} + C \quad (30)$$

$$\int x\sqrt{x^2 \pm a^2} dx = \frac{1}{3}(x^2 \pm a^2)^{3/2} + C \quad (31)$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C \quad (32)$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C \quad (33)$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} + C \quad (34)$$

$$\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2} + C \quad (35)$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{2} x \sqrt{x^2 \pm a^2} \mp \frac{1}{2} a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C \quad (36)$$

$$\int \ln(a^2 x^2 \pm b^2) dx = x \ln(a^2 x^2 \pm b^2) + \frac{2b}{a} \tan^{-1} \frac{ax}{b} - 2x + C \quad (44)$$

$$\int \sqrt{ax^2 + bx + c} dx = \frac{b + 2ax}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| + C \quad (37)$$

$$\int \ln(a^2 - b^2 x^2) dx = x \ln(a^2 - b^2 x^2) + \frac{2a}{b} \tan^{-1} \frac{bx}{a} - 2x + C \quad (45)$$

$$\int x \sqrt{ax^2 + bx + c} dx = \frac{1}{48a^{5/2}} \left(2\sqrt{a} \sqrt{ax^2 + bx + c} - (3b^2 + 2abx + 8a(c + ax^2)) + 3(b^3 - 4abc) \ln \left| b + 2ax + 2\sqrt{a} \sqrt{ax^2 + bx + c} \right| \right) \quad (38)$$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| + C \quad (39)$$

$$\int \ln(ax^2 + bx + c) dx = \frac{1}{a} \sqrt{4ac - b^2} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}} - 2x + \left(\frac{b}{2a} + x \right) \ln(ax^2 + bx + c) + C \quad (46)$$

$$\int \frac{x}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{a} \sqrt{ax^2 + bx + c} + \frac{b}{2a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| + C \quad (40)$$

$$\int x \ln(ax + b) dx = \frac{bx}{2a} - \frac{1}{4} x^2 + \frac{1}{2} \left(x^2 - \frac{b^2}{a^2} \right) \ln(ax + b) + C \quad (47)$$

INTEGRALS WITH LOGARITHMS

$$\int \ln ax dx = x \ln ax - x + C \quad (41)$$

$$\int \frac{\ln ax}{x} dx = \frac{1}{2} (\ln ax)^2 + C \quad (42)$$

$$\int x \ln(a^2 - b^2 x^2) dx = -\frac{1}{2} x^2 + \frac{1}{2} \left(x^2 - \frac{a^2}{b^2} \right) \ln(a^2 - b^2 x^2) + C \quad (48)$$

$$\int \ln(ax + b) dx = \left(x + \frac{b}{a} \right) \ln(ax + b) - x + C, a \neq 0 \quad (43)$$

INTEGRALS WITH EXPONENTIALS

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C \quad (49)$$

$$\int \sqrt{x} e^{ax} dx = \frac{1}{a} \sqrt{x} e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}} \operatorname{erf}(i\sqrt{ax}) + C,$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ (50)

$$\int x e^x dx = (x - 1) e^x + C \quad (51)$$

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2} \right) e^{ax} + C \quad (52)$$

$$\int x^2 e^x dx = (x^2 - 2x + 2) e^x + C \quad (53)$$

$$\int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right) e^{ax} + C \quad (54)$$

$$\int x^3 e^x dx = (x^3 - 3x^2 + 6x - 6) e^x + C \quad (55)$$

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx \quad (56)$$

$$\int x^n e^{ax} dx = \frac{(-1)^n}{a^{n+1}} \Gamma[1 + n, -ax] + C,$$

where $\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$ (57)

$$\int e^{ax^2} dx = -\frac{i\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}(ix\sqrt{a}) + C \quad (58)$$

$$\int e^{-ax^2} dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}(x\sqrt{a}) + C \quad (59)$$

INTEGRALS WITH TRIGONOMETRIC FUNCTIONS

$$\int \sin ax dx = -\frac{1}{a} \cos ax + C \quad (60)$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C \quad (61)$$

$$\int \sin^n ax dx = -\frac{1}{a} \cos ax {}_2F_1 \left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax \right] + C \quad (62)$$

$$\int \sin^3 ax dx = -\frac{3 \cos ax}{4a} + \frac{\cos 3ax}{12a} + C \quad (63)$$

$$\int \cos ax dx = \frac{1}{a} \sin ax + C \quad (64)$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C \quad (65)$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)} \cos^{1+p} ax \times {}_2F_1 \left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax \right] + C \quad (66)$$

$$\int \cos^3 ax dx = \frac{3 \sin ax}{4a} + \frac{\sin 3ax}{12a} + C \quad (67)$$

$$\int \cos ax \sin bxdx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)} + C, a \neq b \quad (68)$$

$$\int \sin^2 ax \cos bxdx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} + C \quad (69)$$

$$\int \sin^2 x \cos x dx = \frac{1}{3} \sin^3 x + C \quad (70)$$

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C \quad (81)$$

$$\int \cos^2 ax \sin bxdx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)} + C \quad (71)$$

$$\int \sec x \tan x dx = \sec x + C \quad (82)$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a} \cos^3 ax + C \quad (72)$$

$$\int \sec^2 x \tan x dx = \frac{1}{2} \sec^2 x + C \quad (83)$$

$$\int \sin^2 ax \cos^2 bxdx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)} + C \quad (73)$$

$$\int \sec^n x \tan x dx = \frac{1}{n} \sec^n x + C, n \neq 0 \quad (84)$$

$$\int \csc x dx = \ln \left| \tan \frac{x}{2} \right| + C = \ln |\csc x - \cot x| + C \quad (85)$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a} + C \quad (74)$$

$$\int \csc^2 ax dx = -\frac{1}{a} \cot ax + C \quad (86)$$

$$\int \tan ax dx = -\frac{1}{a} \ln |\cos ax| + C \quad (75)$$

$$\int \tan^2 ax dx = -x + \frac{1}{a} \tan ax + C \quad (76)$$

$$\int \csc^3 x dx = -\frac{1}{2} \cot x \csc x + \frac{1}{2} \ln |\csc x - \cot x| + C \quad (87)$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1 \left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax \right) + C \quad (77)$$

$$\int \csc^n x \cot x dx = -\frac{1}{n} \csc^n x + C, n \neq 0 \quad (88)$$

$$\int \sec x \csc x dx = \ln |\tan x| + C \quad (89)$$

$$\int \tan^3 ax dx = \frac{1}{a} \ln |\cos ax| + \frac{1}{2a} \sec^2 ax + C \quad (78)$$

PRODUCTS OF TRIGONOMETRIC FUNCTIONS AND MONOMIALS

$$\int \sec x dx = \ln |\sec x + \tan x| + C = 2 \tanh^{-1} \left(\tan \frac{x}{2} \right) + C \quad (79)$$

$$\int x \cos x dx = \cos x + x \sin x + C \quad (90)$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax + C \quad (80)$$

$$\int x \cos ax dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C \quad (91)$$

$$\int x^2 \cos x dx = 2x \cos x + (x^2 - 2) \sin x + C \quad (92)$$

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C \quad (101)$$

$$\int x^2 \cos ax dx = \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax + C \quad (93)$$

$$\int x^n \cos x dx = -\frac{1}{2} (i)^{n+1} [\Gamma(n+1, -ix) + (-1)^n \Gamma(n+1, ix)] + C \quad (94)$$

$$\int e^{bx} \sin ax dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax) + C \quad (102)$$

$$\int x^n \cos ax dx = \frac{1}{2} (ia)^{1-n} [(-1)^n \Gamma(n+1, -iax) - \Gamma(n+1, iax)] + C \quad (95)$$

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) + C \quad (103)$$

$$\int x \sin x dx = -x \cos x + \sin x + C \quad (96)$$

$$\int x \sin ax dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2} + C \quad (97)$$

$$\int e^{bx} \cos ax dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin ax + b \cos ax) + C \quad (104)$$

$$\int x^2 \sin x dx = (2 - x^2) \cos x + 2x \sin x + C \quad (98)$$

$$\int x^2 \sin ax dx = \frac{2 - a^2 x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2} + C \quad (99)$$

$$\int x e^x \sin x dx = \frac{1}{2} e^x (\cos x - x \cos x + x \sin x) + C \quad (105)$$

$$\int x^n \sin x dx = -\frac{1}{2} (i)^n [\Gamma(n+1, -ix) - (-1)^n \Gamma(n+1, -ix)] + C \quad (100)$$

$$\int x e^x \cos x dx = \frac{1}{2} e^x (x \cos x - \sin x + x \sin x) + C \quad (106)$$

PRODUCTS OF TRIGONOMETRIC FUNCTIONS AND EXPONENTIALS

INTEGRALS OF HYPERBOLIC FUNCTIONS

$$\int \tanh bxdx = \frac{1}{a} \ln \cosh ax + C \quad (112)$$

$$\int \cosh axdx = \frac{1}{a} \sinh ax + C \quad (107)$$

$$\int \cos ax \cosh bxdx = \frac{1}{a^2 + b^2} [a \sin ax \cosh bx + b \cos ax \sinh bx] + C \quad (113)$$

$$\int e^{ax} \cosh bxdx = \begin{cases} \frac{e^{ax}}{a^2 - b^2} [a \cosh bx - b \sinh bx] + C & a \neq b \\ \frac{e^{2ax}}{4a} + \frac{x}{2} + C & a = b \end{cases} \quad (108)$$

$$\int \cos ax \sinh bxdx = \frac{1}{a^2 + b^2} [b \cos ax \cosh bx + a \sin ax \sinh bx] + C \quad (114)$$

$$\int \sinh axdx = \frac{1}{a} \cosh ax + C \quad (109)$$

$$\int \sin ax \cosh bxdx = \frac{1}{a^2 + b^2} [-a \cos ax \cosh bx + b \sin ax \sinh bx] + C \quad (115)$$

$$\int e^{ax} \sinh bxdx = \begin{cases} \frac{e^{ax}}{a^2 - b^2} [-b \cosh bx + a \sinh bx] + C & a \neq b \\ \frac{e^{2ax}}{4a} - \frac{x}{2} + C & a = b \end{cases} \quad (110)$$

$$\int \sin ax \sinh bxdx = \frac{1}{a^2 + b^2} [b \cosh bx \sin ax - a \cos ax \sinh bx] + C \quad (116)$$

$$\int e^{ax} \tanh bxdx = \begin{cases} \frac{e^{(a+2b)x}}{(a+2b)} {}_2F_1 \left[1 + \frac{a}{2b}, 1, 2 + \frac{a}{2b}, -e^{2bx} \right] - \frac{1}{a} e^{ax} {}_2F_1 \left[\frac{a}{2b}, 1, 1E, -e^{2bx} \right] + C & a \neq b \\ \frac{e^{ax} - 2 \tan^{-1}[e^{ax}]}{a} + C & a = b \end{cases} \quad (111)$$

$$\int \sinh ax \cosh axdx = \frac{1}{4a} [-2ax + \sinh 2ax] + C \quad (117)$$

$$\int \sinh ax \cosh bxdx = \frac{1}{b^2 - a^2} [b \cosh bx \sinh ax - a \cosh ax \sinh bx] + C \quad (118)$$