MATH 234 - WINTER QUARTER 2012

ODE-PALOOZA SOLUTIONS!!!!

Do the following problems on your own paper. You must show your work to get credit for your answers.

1. Solve the following problems. If initial conditions are given, solve the initial value problem. If no initial conditions are given, find the general solution. *If you choose series methods, please find the first three non-zero terms in the sum.*

   (a) $2y'' + 4y' + 2y = xe^x$

   1. Find homogeneous solution
   $$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}$$
   2. Find particular solution via the method of undetermined coefficients
   $$y_p(x) = \frac{1}{8} e^x (x-1)$$
   $$\boxed{y(x) = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{8} e^x (x-1)}$$

   (b) $x^2 y'' + xy' - y = 0$

   First, change the ODE so that the last term is $-y$ and not $+y$. This is an Euler type ODE. Thus, assume $y = x^r$ and find r. This gives
   $$r(r-1) + r - 1 = 0 \quad \rightarrow \quad r = \pm 1$$
   Thus,
   $$\boxed{y(x) = c_1 x^1 + c_2 x^{-1}}$$

   (c) $2x + y^2 + 2xyy' = 0$

   First order, nonlinear, not separable. But it is exact. $f_x = 2x + y^2$ and $f_y = 2xy$. Does $f_{xy} = f_{yx}$? YES!
   $$\boxed{x^2 + xy^2 = c}$$

   (d) $y'(t) = y(t) + 4z(t) + e^t$
   $z'(t) = 5z(t) - y(t)$

Write this as a system of first order equations.

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

Now, find the solution using any method you like.

(e) $\dfrac{dz}{dt} = z^2 \sin(t)$

This ODE is separable.

$$\int z^{-2}\, dz = \int \sin(t)\, dt$$

$$\boxed{\text{The implicit solution } -z^{-1} = -\cos(t) + c}$$

(f) $y' = y\cos(x) + y - 1$

First Order, linear. Write in std. form as

$$y' - (1 + \cos(x))y = -1$$

Try linear integrating function to get

$$\frac{d}{dx}\left(ye^{-\int 1+\cos(x)\, dx}\right) = -e^{-\int 1+\cos(x)\, dx}$$

Then, the solution (up to some integrals we can't evaluate) is given by

$$\boxed{y(x) = e^{\int 1+\cos(x)\, dx}\left(-\int e^{-\int 1+\cos(x)\, dx} + c\right)}$$

(g) $y'' - y = \cos(t) - u_\pi(t)\cos(t-\pi)$ with $y(0) = 0, y'(0) = 0$

Second order, linear, constant coefficient, USE LAPLACE!

$$\boxed{\frac{1}{4}\left(e^t + e^{-t}\right) - \frac{1}{2}\cos(t) - u_\pi(t)\left(\frac{1}{4}\left(e^{t-\pi} + e^{-t+\pi}\right) - \frac{1}{2}\cos(t-\pi)\right)}$$

(h) $y'' + 2y' + y = 0$

Simple!

$$\boxed{y(x) = c_1 e^{-x} - c_2 x e^{-x}}$$

(i) $\vec{\mathbf{y}} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 3 \\ 0 & 0 & -1 \end{bmatrix} \vec{\mathbf{y}}$

Using 3×3 eigenvalue/eigenvector methods, we find

$$\boxed{\vec{\mathbf{y}} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} -5 \\ 3 \\ 4 \end{bmatrix} e^{-t}}$$

(j) $(4 - x^2)y'' + y = 0$ near $x = 0$

Using a series solution, we find

$$\boxed{y(x) = a_0 + a_1 x - \frac{a_0}{8}x^2 - \frac{a_1}{24}x^3 + \ldots}$$

(k) $(1 + x^2)y' + 4xy - \dfrac{1}{1 + x^2} = 0$

First order, linear! Must put in std. form. See Exam #1 solutions for full details.

$$\boxed{y(x) = \frac{x}{(1 + x^2)^2} + \frac{c}{(1 + x^2)^2}}$$

(l) $-(x^2 + 2xy)\dfrac{dy}{dx} = 2xy + y^2 + 1$

First order, nonlinear, not separable. Is it exact? Rewrite as

$$2xy + y^2 + 1 + (x^2 + 2xy)\frac{dy}{dx} = 0$$

and check; yup! It's exact.

$$\boxed{x^2 y + xy^2 + x = c}$$

(m) $y'' + 8y' + 16y = 3xe^{-4x} + x$

1. Find homogeneous solution
$$y_h(x) = c_1 e^{-4} + c_2 x e^{-4}$$
2. Find particular solution via the method of undetermined coefficients, but you must be careful. The initial guess would be $y_p(x) = (Ax + B)e^{-4x} + Cx + D$, but the first two terms are precisely the homogeneous solution. Thus, you should multiply that part (and only that part) of the guess by x until the homogeneous terms are no longer visible. This gives a guess of
$$y_p(x) = (Ax^3 + Bx^2)e^{-4x} + Cx + D$$
. Plugging it all in and collecting like terms (AKA, have mathematica do it for me), we get
$$\boxed{y(x) = c_1 e^{-4x} + c_2 x e^{-4x} + \frac{1}{2} x e^{-4x} + \frac{1}{16} x - \frac{1}{32}}$$

(n) $y''' + 2y'' + y' = 0$

Try anything! BUT, since it's constant coefficient, and homogeneous, just try $y = e^{\lambda x}$. You find
$$\lambda^3 + 2\lambda^2 + \lambda = 0$$
which can be factored to find
$$\lambda(\lambda + 1)^2 = 0$$
This yields the following final answer
$$\boxed{y(x) = c_1 + c_2 e^{-x} + c_3 x e^{-x}}$$