

5

part a

$$y'' + 4y' + 4y = 0$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda + 2)^2 = 0 \rightarrow \lambda = -2, \lambda = -2$$

$$\rightarrow y = c_1 e^{-2t} + c_2 x e^{-2t}$$

$$y(0) = 1 \rightarrow c_1 = 1$$

$$y'(0) = 0 \rightarrow -2 + c_2 = 0 \rightarrow c_2 = 2$$

$$y = e^{-2t} + 2te^{-2t}$$

part b

$$\text{let } \left. \begin{matrix} u_1 = y \\ u_2 = y' \end{matrix} \right\} \rightarrow \begin{matrix} u_1' = y' = u_2 \\ u_2' = y'' = -4u_1 - 4u_2 \end{matrix}$$

$$\rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \hat{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

part c

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \quad A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -4 & -4 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$
$$\hookrightarrow \lambda^2 + 4\lambda + 4 = 0$$

$$\lambda = -2$$

→ repeated e-values

$$\lambda_1 = -2$$

$$(A - \lambda I) \hat{\xi}_1 = 0$$

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} \xi_{11} \\ \xi_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$$

$$2\xi_{11} + \xi_{12} = 0$$

$$\xi_{11} = 1, \xi_{12} = -2$$

$$\rightarrow \hat{\xi}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\lambda_2 = -2$$

"eigenvector" $\hat{\eta}$

$$(A - \lambda I) \hat{\eta} = \hat{\xi}_1$$

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} \eta_{11} \\ \eta_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\left. \begin{array}{l} 2\eta_{11} + \eta_{12} = 1 \\ -4\eta_{11} - 2\eta_{12} = -2 \end{array} \right\} \rightarrow \eta_{11} = 1, \eta_{12} = -1$$

$$\eta_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Fundamental sols

$$\rightarrow \hat{x}_1 = \hat{\xi}_1 e^{\lambda t}$$

$$\hat{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}$$

$$\hat{x}_2 = \hat{\xi}_1 t e^{\lambda t} + \hat{\eta} e^{\lambda t}$$

$$\hat{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} t e^{-2t} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

General Sol

$$\hat{x} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} t e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

Solving for $c_1 + c_2$ using the initial conditions, we have...

$$\hat{x}(t) = \begin{bmatrix} c_1 e^{-2t} + c_2 (te^{-2t} + e^{-2t}) \\ c_1 (-2)e^{-2t} + c_2 (-2te^{-2t} - e^{-2t}) \end{bmatrix}$$

$$\begin{aligned} c_1 + c_2 &= 1 \\ -2c_1 - c_2 &= 0 \end{aligned} \rightarrow \begin{aligned} c_2 &= -2c_1 \\ -c_1 &= 1 \rightarrow c_1 = -1, c_2 = 2 \end{aligned}$$

$$\hat{x}(t) = \begin{bmatrix} e^{-2t} + 2te^{-2t} \\ -4te^{-2t} \end{bmatrix}$$

Since $y(t) = x_1(t)$, we have $y(t) = e^{-2t} + 2te^{-2t}$
 also, note that since $x_2(t) = y'$, we get the derivative of y in the second entry of the vector solution.

#6

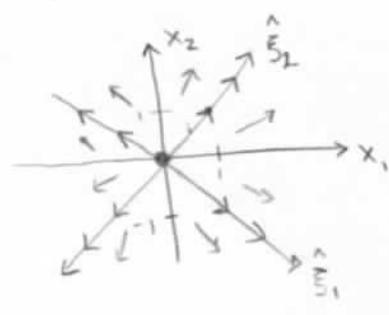
$$\left. \begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_3 &= y'' \\ x_4 &= y''' \end{aligned} \right\} \rightarrow \begin{aligned} x_1 &= y \\ x_2 &= y' = x_2 \\ x_3 &= y'' = x_3 \\ x_4 &= y''' = -4y''' + 3y - y \\ &= -4x_4 + 2x_1 \end{aligned} \quad \hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\hat{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & -4 \end{bmatrix} \hat{x}, \quad \hat{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

#7

part a

$$\lambda_1 = \frac{1}{2}, \hat{\xi}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \lambda_2 = 2, \hat{\xi}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



→ since both λ_1, λ_2 are > 0 , the origin $(0,0)$ is unstable

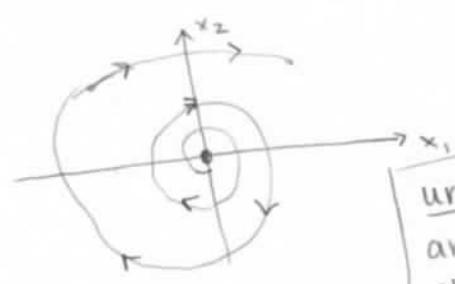
part b

$$\lambda_1 = 1+i, \hat{\xi}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\rightarrow \hat{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(1+i)t} = \begin{bmatrix} -i(e^t)(\cos(t) + i\sin(t)) \\ e^t(\cos(t) + i\sin(t)) \end{bmatrix}$$

$$= e^t \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + i e^t \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$$

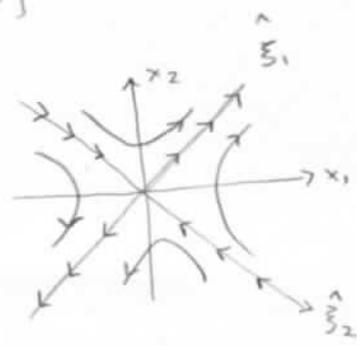
real part imag. part



unstable spiral since $\text{re}(\lambda) > 0$ (e^t -term) and the parametric eqns yield clockwise motion.

part c

$$\lambda_1 = \sqrt{122}, \hat{\xi}_1 = \begin{bmatrix} 11 \\ 1 + \sqrt{122} \end{bmatrix}, \lambda_2 = -\sqrt{122}, \hat{\xi}_2 = \begin{bmatrix} 11 \\ 1 - \sqrt{122} \end{bmatrix}$$



unstable saddle

#8 from earlier, we know the e-vals/vectors

$$\text{homog sol} \rightarrow \hat{x}(t) = c_1 \begin{bmatrix} 11 \\ 1+\sqrt{122} \end{bmatrix} e^{\sqrt{122}t} + c_2 \begin{bmatrix} 11 \\ 1-\sqrt{122} \end{bmatrix} e^{-\sqrt{122}t}$$

neither of the funda. sols are repeated through the forcing term, so we can use undetermined coeffs, Laplace transforms, or variation of params in a straight forward fashion.

Variation of params

$$* \hat{x}(t) = \underbrace{\varphi(t)}_{\substack{\text{fundamental} \\ \text{matrix}}} \hat{c} + \underbrace{\varphi(t)}_{\substack{\text{vector of constants}}} \int \varphi^{-1}(t)g(t) dt$$

$$\rightarrow \varphi(t) = \begin{bmatrix} 11 e^{\sqrt{122}t} & 11 e^{-\sqrt{122}t} \\ (1+\sqrt{122}) e^{\sqrt{122}t} & (1-\sqrt{122}) e^{\sqrt{122}t} \end{bmatrix}$$

$$\varphi^{-1}(t) = \frac{1}{-22\sqrt{122}} \begin{bmatrix} (1-\sqrt{122}) e^{-\sqrt{122}t} & -11 e^{-\sqrt{122}t} \\ -(1+\sqrt{122}) e^{-\sqrt{122}t} & (11) e^{\sqrt{122}t} \end{bmatrix}$$

$$g(t) = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix} \rightarrow \text{plug into } (*) \text{ and solve...}$$