Math 234 - Final Exam Study Guide

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Preliminary Material

In the first few lectures, we covered some basic definitions and classifications of differential equations. Here are a few of the classifications:

Definition (Order of an ODE)

The order of a differential equation is determined by the order of the highest derivative of the equation. For example,

$$\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} \cdot \frac{d^3y}{dx^3} = \cos(x)$$

is a forth-order ODE because the highest derivative is a forth order derivative.

Definition (Linear vs. Nonlinear ODE)

A differential equation is linear if the dependent variable, as well as all of it's derivatives, are linear. For example, a linear ODE can be written as

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_0(x)y = f(x)$$

This means that $y'' + 3x^3y' + \cos(x)y = e^x$ is linear since y and all of its derivatives show up linearly.

On the other hand, $y'y = 1$ is nonlinear because of the $y'y$ term.

First Order Differential Equations

The most general first order ODE can be written in the form

$$y' = f(x, y).$$

We discussed several ways to analyze first order differential equations without solving. These methods include

- Examining the direction field - Let $y' = f(x, y)$. To draw the direction field,

1. Choose a point in the x-y plane (x_0, y_0), and evaluate y' at this point.
2. At the point (x_0, y_0), draw a small arrow with the slope found in part (1).
3. Repeat this process for a lot of points (x_0, y_0). The plot that you obtain will give you a good idea of how solutions behave. *NOTE: You can use the dfield applet found at* http://math.rice.edu/ dfield/dfpp.html *to automatically generate this plot.*

- If $y' = f(y)$ (there is no dependence on x), we can plot y vs. y'. If y' is positive, the solution $y(x)$ is increasing, and if y' is negative, the solution is decreasing. This also gives us information regarding the behavior of solutions without solving the differential equation.

In terms of analytic solutions, we discussed several ways to solve first order differential equations. These techniques or approaches are summarized below.

Separable Differential Equations

A first order differential equation is *separable* if we can write it in the form

$$\frac{dy}{dx} = f(x) \cdot g(y) \quad \Longrightarrow \quad \int \frac{dy}{g(y)} = \int f(x)\ dx + c$$

In other words, if we can separate the x's and the y's, we can solve the equation by integrating each side of the equation.

First Order Linear Differential Equations and Integrating Factors

The most general form of a first order linear differential equation is

$$y' + p(x)y = g(x)$$

Existence and Uniqueness

Theorem

Consider the initial value problem

$$y' + p(x)y' = g(x), \qquad y(x_0) = y_0,$$

where $p(x)$ and $g(x)$ are continuous on an open interval I that also contains x_0[1]. Then, there exists exactly one solution $y = \phi(x)$ of this problem and the solution exists throughout the interval I.

It's important to note the three things that this theorem says:

1. The initial value problem has *a* solution; in other words, a solution *exists*.

[1] I is just a range of x values where the functions p and g are well behaved.

2. The initial value problem has *only one* solution; in other words, the solution is *unique*
3. The solution $\phi(x)$ is defined *throughout* the interval I where the coefficients are continuous and is at least once differentiable there.

Integrating Factor (Linear First Order ODE)

The integrating factor for a first order linear ODE of the form

$$y' + p(x)y = g(x)$$

is given by

$$\mu(x) = e^{\int p(x)\ dx}$$

General Solution (Linear First Order ODE)

The general solution of a first order linear ODE of the form

$$y' + p(x)y = q(x)$$

is given by

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)g(x)\ dx + c\right) \quad \text{or} \quad y = \frac{1}{\mu(x)}\left(\int_{x_0}^{x} \mu(x)g(x)\ dx + \mu(x_0)y_0\right)$$

Exact Differential Equations and Integrating Factors

Exact Equation

An ODE of the form

$$N(x,y)y' + M(x,y) = 0,$$

is exact if and only if

$$(N(x,y))_x = (M(x,y))_y.$$

If the ODE is exact, then the solution in implicit form is $f(x,y) = c$ where $f(x,y)$ can be found by solving

$$N(x,y) = f_y \quad \text{and} \quad M(x,y) = f_x$$

Integrating Factors (Nonlinear First Order ODE)

If an ode of the form

$$N(x,y)y' + M(x,y) = 0,$$

is not exact, the goal is to find a function $\mu(x,y)$ so that when we multiply the ODE by $\mu(x,y)$ the equation becomes exact. In other words, can we find a $\mu(x,y)$ so that

$$\mu(x,y)N(x,y)y' + \mu(x,y)M(x,y) = 0,$$

is exact. This means that

$$(\mu(x,y)N(x,y))_x = (\mu(x,y)M(x,y))_y.$$

Typically, we assume that μ is a function of x or y ONLY. In doing so, we hopefully can find a solution for the function μ by solving the equation that results from $(\mu N)_x = (\mu M)_y$.

For example, if μ is a function of x only, then the above condition simplifies to

$$(\mu N)_x = (\mu M)_y \quad \Longrightarrow \quad \mu_x N + \mu N_x = \mu M_y$$

We can write this as a first order ODE for μ as

$$\mu_x = \mu \frac{(M_y - N_x)}{N} \tag{1}$$

and if $\frac{(M_y - N_x)}{N}$ depends only on x, we can solve this ODE for the integrating factor μ via separation of variables.

What would Equation (1) be if we assumed that μ was a function of only y? *Hint: expand* $(\mu N)_x = (\mu M)_y$ *using the product rule for derivatives remembering that* μ *does not depend on* x *so that* $\mu_x = 0$.

Second Order Differential Equations (Homogeneous)

Constant Coefficient Problems

Consider the Linear 2nd order Constant Coefficient Homogeneous ODE (whew... that's a mouthful).

$$ay'' + by' + cy = 0$$

We can solve this differential equation with the following steps:

1. Assume that the solution is of the form $y = e^{\lambda x}$. Substitute this solution form into the differential equation to determine the characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

2. Solve the characteristic equation $a\lambda^2 + b\lambda + c = 0$ to find the roots or λ values. This equation has two roots, λ_1 and λ_2.
3. Write down the fundamental solutions:
   (a) λ_1 and λ_2 are both real and unique ($\lambda_1 \neq \lambda_2$). In this case, the two fundamental solutions are

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}$$

   (b) $\lambda_1 = \lambda_2$ (the case of real, repeated roots). Then the two fundamental solutions are

$$y_1 = e^{\lambda_1 x}, \quad y_2 = xe^{\lambda_1 x}$$

(c) λ_1 and λ_2 are complex conjugates ($\lambda_1 = \alpha + i\beta$, and $\lambda_2 = \alpha - i\beta$). In this case, the two fundamental solutions are

$$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x),$$

4. Once you know the two fundamental solutions, you can write down the general solution

$$y = c_1 y_1 + c_2 y_2.$$

Euler Equation Problems

Consider a 2nd order ODE of the Euler type in the form

$$x^2 y'' + \alpha x y' + \beta y = 0$$

We can solve this differential equation with the following steps:

1. Assume that the solution is of the form $y = x^s$. Substitute this solution form into the differential equation to determine the indicial equation

$$s(s-1) + \alpha s + \beta = 0 \quad \Longrightarrow \quad s^2 + (\alpha - 1)s + \beta = 0$$

2. Solve the indicial equation $s^2 + (\alpha - 1)s + \beta = 0$ to find the roots or s values. This equation has two roots, s_1 and s_2.

3. Write down the fundamental solutions:

   (a) s_1 and s_2 are both real and unique ($s_1 \neq s_2$). In this case, the two fundamental solutions are

   $$y_1 = x^{s_1}, \quad y_2 = x^{s_2}$$

   (b) $s_1 = s_2$ (the case of real, repeated roots). Then the two fundamental solutions are

   $$y_1 = x^{s_1}, \quad y_2 = x^{s_1}\ln(x)$$

   (c) s_1 and s_2 are complex conjugates ($s_1 = \eta + i\mu$, and $\lambda_2 = \eta - i\mu$). In this case, the two fundamental solutions are

   $$y_1 = x^{\eta}\cos(\mu \ln(x)), \quad y_2 = x^{\eta}\sin(\mu \ln(x))$$

4. Once you know the two fundamental solutions, you can write down the general solution

$$y = c_1 y_1 + c_2 y_2.$$

Existence

Theorem

Consider the initial value problem

$$y'' + p(x)y' + q(x)y = g(x), \qquad y(x_0) = y_0,\ y'(x_0) = y_0',$$

where $p(x)$, $q(x)$ and $g(x)$ are continuous on an open interval I that also contains ${x_0}^2$. Then, there exists exactly one solution $y = \phi(x)$ of this problem and the solution exists throughout the interval I.

It's important to note the three things that this theorem says:

1. The initial value problem has *a* solution; in other words, a solution *exists*.
2. The initial value problem has *only one* solution; in other words, the solution is *unique*
3. The solution $\phi(x)$ is defined *throughout* the interval I where the coefficients are continuous and is at least twice differentiable there.

Example:

Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)\frac{d^2y}{dt^2} + t\frac{dy}{dt} - (t+3)y = 0, \qquad y(1) = 2, \qquad \left.\frac{dy}{dt}\right|_{t=1} = 1$$

In this problem, if we write it in the form where the coefficient of the second derivative term is one, we find that $p(t) = 1/(t-3)$, $q(t) = -(t+3)/(t^2 - 3t)$, and $g(t) = 0$. The only points of discontinuity of the coefficients are at $t = 0$ and $t = 3$. Therefore, the longest open interval containing the initial point $t = 1$ in which all of the coefficients are continuous is $I = 0 < t < 3$. Thus, this is the longest interval in which our theorem guarantees that a solution exists.

The Wronskian, Abel's Theorem, and Reduction of Order

Theorem

Two functions $f(x)$ and $g(x)$ are linearly dependent if their Wronskian

$$W(f,g)(x) = f(x)g'(x) - f'(x)g(x) = 0.$$

We can also calculate the the Wronskian as the determinant of the following matrix:

$$W(f,g)(x) = \det \begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} = f(x)g'(x) - f'(x)g(x)$$

Theorem (Abels theorem)

Let y_1 and y_2 be any two fundamental solutions of

$$y'' + p(x)y' + q(x)y = 0,$$

then

$$W(y_1, y_2) = ce^{-\int p(x)dx},$$

[2] I is just a range of x values where the functions p, q, and g are well behaved.

where c is a constant. ★

Formula (Reduction of Order)

If you are given one fundamental solution (y_1) to the ODE

$$y'' + p(x)y' + q(x)y = 0,$$

you can determine the second fundamental solution via Abel's theorem using the following formula:

$$y_2 = y_1 \int \frac{W}{y_1^2} dx.$$

*Recall **how** we derived this in class.*

Second Order Differential Equations (Homogeneous)

A linear second-order nonhomogeneous ODE is of the form

$$y'' + p(x)y' + q(x)y = g(x)$$

We discussed how the solution $y(x)$ consisted of two parts:

1. The homogeneous solution y_h. This solution solves the problem $y'' + p(x)y' + q(x)y = 0$.
2. The particular solution y_p. This solution solves the problem $y'' + p(x)y' + q(x)y = g(x)$.

There are two different methods to solve the nonhomogeneous problem: (1) the method of undetermined coefficients, and (2) variation of parameters.

Method of Undetermined Coefficients

If the differential equation is of the form

$$y'' + p(x)y' + q(x)y = g(x)$$

we can *guess* the form of the particular solution using the following rules:

Rule Name	If $g(x)$ is of the form...	Guess $y_p(x)$ of the form...
Constant	c	A
Exponential	$\alpha e^{\kappa x}$	$Ae^{\kappa x}$
Polynomial	$\alpha_N x^N + \ldots + \alpha_1 x + \alpha_0$	$A_N x^n + \ldots + A_1 x + A_0$
Exp-Poly	$e^{\kappa x}\left(\alpha_N x^N + \ldots + \alpha_1 x + \alpha_0\right)$	$e^{\kappa x}\left(A_N x^n + \ldots + A_1 x + A_0\right)$
Cosine-Sine	$\alpha\cos(\omega x) + \beta\sin(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$
Poly Cosine-Sine	$P_n(x)\cos(\omega x) + Q_m(x)\sin(\omega x)$	$S_N(x)\cos(\omega x) + T_N(x)\sin(\omega x)$
Poly-Exp Cosine-Sine	$e^{\kappa x}\left(P_n(x)\cos(\omega x) + Q_m(x)\sin(\omega x)\right)$	$e^{\kappa x}\left(S_N(x)\cos(\omega x) + T_N(x)\sin(\omega x)\right)$

So, in general, we use our seven rules to give an initial guess for the form of y_P. Next, if any term in this guess appears in the homogeneous solution, we multiply the entire guess by x. Now, we check if any of the new terms still appear in the homogeneous solution. If so, we multiply by x again, and so on.

Variation of Parameters

Theorem (Variation of Parameters)

If the functions p, q and g are continuous on an open interval I, and if the functions y_1 and y_2 are linearly independent solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

then a particular solution of

$$y'' + p(x)y' + q(x)y = g(x),$$

is

$$y_p(x) = -y_1(x)\int \frac{y_2(x)g(x)}{W(y_1,y_2)(x)}\,dx + y_2(x)\int \frac{y_1(x)g(x)}{W(y_1,y_2)(x)}\,dx$$

The general solution is

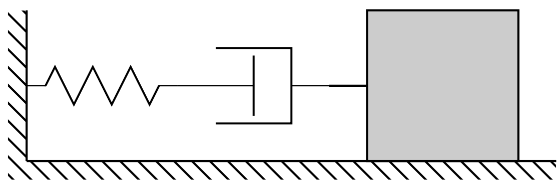
$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

Mechanical Vibrations and Forcing

There are two main models of mechanical forcing that we discussed in class: (1) the mass-spring-damper system, and (2) the Resistor-Capacitor-Inductor Circuit. You should know the basic mathematical model for both systems, how to solve them, and important properties of the solutions.

The Mass-Spring-Damper System (MSD)

Consider the mechanical system shown in the figure below:



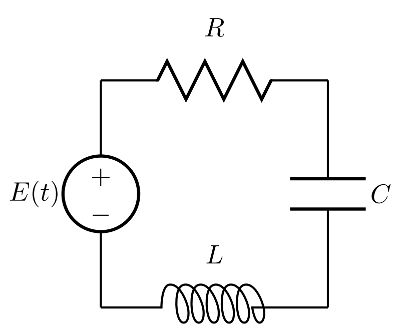
The differential equation that models the mass-spring-damper system is

$$mx'' + bx' + kx = F_e(t)$$

where m is the mass of the object, b is the damping constant, k is the spring constant, and $F_e(t)$ is the external forcing applied to the system.

The Resistor-Capacitor-Inductor Circuit (RCL)

Consider the electrical circuit shown in the figure below:



The differential equation that models the RCL system is

$$LI'' + RI' + \frac{1}{C}I = E'(t)$$

where $E(t)$ is the voltage source, $E'(t)$ is the derivative of the power source, L is the inductance, R is the resistance, and C is the capacitance. $I(t)$ represents the current in the circuit at time t.

DEFINITION:

Consider a differential equation of the form

$$y'' + \omega_0^2 y = g(t)$$

The system will exhibit **resonance** when it is forced at the same frequency as the homogeneous solution. In other words, if $g(t)$ has a term of the form $\sin(\omega_0 t)$ or $\cos(\omega_0 t)$, the system will exhibit resonance since the homogeneous solution is of the form

$$y_h(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

This mean that the solution will grow linearly in time.

DEFINITION:

Consider a differential equation of the form

$$y'' + \omega_0^2 y = g(t)$$

The system will exhibit **beats** when the homogeneous system oscillates at a different frequency then the forcing function. In other words, if $g(t)$ has a term of the form $\sin(\omega t)$ or $\cos(\omega t)$ where $\omega_0 \neq \omega$, the system will exhibit resonance since beats or modulations since the solution will contain two frequencies.

DEFINITION:

If you have found your solution in the form

$$x(t) = e^{\alpha t} \left(c_1 \cos(\omega t) + c_2 \sin(\omega t) \right)$$

the **amplitude phase form** of the solution is given by

$$x(t) = Ae^{\alpha t} \cos(\omega t + \varphi), \quad \text{where } A = \sqrt{c_1^2 + c_2^2}, \quad \varphi = \tan^{-1}\left(\frac{c_1}{c_2}\right)$$

The **solution envelope** is given by $\pm Ae^{\alpha t}$

Higher Order Linear Constant Coefficient ODEs

Consider the linear n-th order constant-coefficient homogeneous ODE:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = g(x)$$

where the a_j's are constant coefficients.

The Homogeneous Problem

If we want to find the solution to the homogeneous problem, we set $g(x) = 0$. We can solve this differential equation with the following steps:

1. Assume that the solution is of the form $y = e^{\lambda x}$. Substitute this solution form into the differential equation to determine the characteristic equation

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$$

2. Solve the characteristic equation (also referred to as the characteristic polynomial) $a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$ to find the roots or λ values. Since this is an n-degree polynomial for λ, there are n roots, λ_1, λ_2, $\ldots$ λ_n.

3. Write down the fundamental solutions:

   Real and Unique Roots If the roots are real and not equal, the fundamental solutions will take the form:

$$e^{\lambda_j x}$$

   Repeated Roots If the roots of the characteristic equation are repeated (say with multiplicity s), the fundamental solutions will take the form:

$$e^{\lambda_1 x}, \quad xe^{\lambda_1 x}, \quad \ldots, \quad x^{s-1}e^{\lambda_1 x}$$

   Complex Roots If the roots of the characteristic equation are complex, the must occur in complex conjugate pairs ($\lambda = \alpha \pm i\beta$). The fundamental solutions will take the form:

$$e^{\alpha x}\cos(\beta x), \quad e^{\alpha x}\sin(\beta x),$$

4. Once you know all of the fundamental solutions, you can write down the general solution

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$$

Example

Find the general solution of

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 16y = 0$$

Solution

Begin by substituting the solution form $y = e^{\lambda x}$. This yields

$$y = e^{\lambda x}, \quad \frac{dy}{dx} = \lambda e^{\lambda x}, \quad \frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}, \quad \frac{d^3y}{dx^3} = \lambda^3 e^{\lambda x}$$

Plugging this in, we find that the characteristic equation becomes

$$\lambda^3 - 4\lambda^2 + 4\lambda - 16 = 0$$

We can solve this equation by factoring:

$$\begin{aligned} \lambda^3 - 4\lambda^2 + 4\lambda - 16 &= 0 \\ \lambda^2(\lambda - 4) + 4(\lambda - 4) &= 0 \\ (\lambda^2 + 4)(\lambda - 4) &= 0 \end{aligned}$$

This implies that $\lambda^2 + 4 = 0$ or $\lambda - 4 = 0$. Thus, the roots are $\lambda_1 = 2i$, $\lambda_2 = -2i$, and $\lambda_3 = 4$. Now we can write down the general solution using the form of the roots:

$$\boxed{y(x) = c_1 \cos(2x) + c_2 \sin(2x) + c_3 e^{4x}}$$

Laplace Transforms

Definition:

The **Laplace transform** of a function $f(t)$ is given by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\ dt$$

On the exam you will be given the same table of transforms that you were given in class.

Theorem

Assume that $f(t)$ is an n-times differentiable function whose Laplace transform is given by $F(s)$. Then: The Laplace transform of $f'(t)$ is given by

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

The Laplace transform of $f''(t)$ is given by

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

The Laplace transform of $f^{(n)}(t)$ is given by

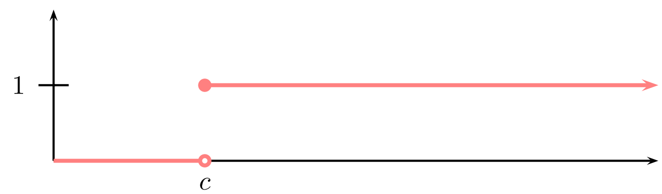
$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^nF(s) - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

The Unit Step Function

The unit step function $u_c(t)$ is defined as the piecewise function

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

The graph of $u_c(t)$ is given in the figure below.



The Impulse Function

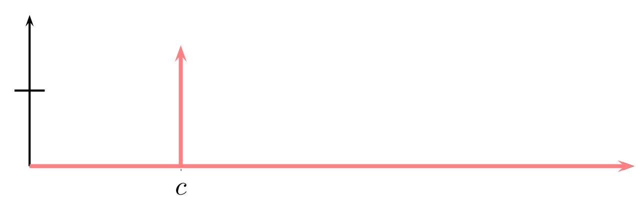
The impulse function $\delta(t)$ is defined by the following properties:

$$\delta(t-c) = 0 \text{ if } t \neq c \int_{-\infty}^{\infty} \delta(t)\ dt = 1$$

Note that

$$\int_{-\infty}^{\infty} \delta(t-c)f(t)\ dt = f(c), \qquad \int_{0}^{\infty} \delta(t-c)f(t)\ dt = f(c),\ c > 0$$

In *theory*, we can imagine that the graph of $\delta(t-c)$ would look like the one given in the figure below:



Definition:

The **convolution** of two functions $f(t)$ and $g(t)$ is given by

$$f * g = \int_0^t f(t-\tau)g(\tau)\ d\tau$$

There are several important perperties of the convolution:

1. $f * g = g * f$ (commutative property)
2. $f * (g_1 + g_2) = f * g_1 + f * g_2$ (distributive property)
3. $f * (g * h) = (f * g) * h$ (associative property)

Theorem

The Inverse Transform of the Product of Two Transforms: If $\mathcal{L}\{f(t)\} = F(s)$ *and* $\mathcal{L}\{g(t)\} = G(s)$ *and* $H(s) = F(s) \cdot G(s)$, *then the inverse transform of* $H(s)$ *is given by*

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = f(t) * g(t)$$

Example

Finding the Inverse Laplace Transform (via Partial Fractions)
Find the inverse transform of

$$H(s) = \frac{1}{s^2 + 3s}$$

Solution

First we start by using a partial fraction expansion on the function

$$H(s) = \frac{1}{s^2 + 3s} = \frac{A}{s} + \frac{B}{s+3}$$

Solving for A and B, we find

$$H(s) = \frac{1/3}{s} - \frac{1/3}{s+3}$$

Using the Laplace table, we find

$$\boxed{h(t) = \frac{1}{3} - \frac{1}{3}e^{-3t}}$$

Example

Finding the Inverse Laplace Transform (via the Convolution Theorem)
Find the inverse transform of

$$H(s) = \frac{1}{s^2 + 3s}$$

Solution

First we start by thinking of $H(s)$ as the product of two transforms $F(s)$ and $G(s)$ where

$$H(s) = \frac{1}{s} \cdot \frac{1}{s+3} = F(s) \cdot G(s)$$

By using the table, we see that $f(t) = 1$ and $g(t) = e^{-3t}$. Using the convolution theorem, we find that

$$\begin{aligned} h(t) &= f * g \\ &= \int_0^t 1 \cdot e^{-3\tau}\ d\tau \\ &= -\frac{1}{3} e^{-3\tau} \|_0^t \\ &= -\frac{1}{3}\left(e^{-3t} - 1\right) \end{aligned}$$

Notice that we find the same answer using the convolution as we did using partial fractions.

Series Solutions

Definition:

Consider the general second order differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

- A point x_0 is a **regular point** of the differential equation if $P(x_0) \neq 0$. This implies that there is a solution in a neighborhood of x_0 which can be expressed in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

- A point x_0 is a **regular singular point** of the differential equation if all three of the following conditions are satisfied:

  1. $P(x_0) = 0$
  2. $\lim_{x \to x_0} (x - x_0)\dfrac{Q(x)}{P(x)}$ exists
  3. $\lim_{x \to x_0} (x - x_0)^2 \dfrac{R(x)}{P(x)}$ exists

  This implies that there is a solution near x_0 which can be expressed in the form

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Systems of Differential Equations

Homogeneous Systems of First Order Differential Equations

You should know the following:

1. How to write a differential equation as a system of first-order differential equations
2. How to find the fundamental solutions of a system of the form $\mathbf{x}' = A\mathbf{x}$ where A is a constant $n \times n$ matrix and $\mathbf{x}$ is an $n \times 1$ vector of functions.
3. How to test if a set of solutions $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$ is linearly independent.

The Wronskian for Systems of First-Order Differential Equations

If $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$ are vector functions, the Wronskian is defined as

$$W(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \det \begin{pmatrix} \mathbf{x}_1 & \vdots & \mathbf{x}_2 & \vdots & \ldots & \vdots \mathbf{x}_n \end{pmatrix}$$

Theorem

Abel's Theorem for Systems of First-Order Differential Equations
Consider a general Matrix ODE of the form

$$\mathbf{x}(t) = A(t)\mathbf{x}$$

where $A(t)$ is an $n \times n$ matrix of functions. Abel's theorem states that the Wronskian of the n fundamental solutions $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$ is given by

$$W(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = c \exp \left[\int \mathit{Trace} A(t)\ dt \right]$$

Solving Linear Homogeneous Systems of First Order Differential Equations with Constant Coefficients

Example

Writing an ODE as a First Order System
Consider the initial-value problem

$$y'' - 5y' + 6y = 0, \qquad y(0) = 1, \quad y'(0) = 0$$

Write this second order differential equation as a system of first-order differential equations.

Solution

Start by letting $x_1(t) = y(t)$ and $x_2(t) = y'(t)$. Taking a derivative of each equation, we find

$$x_1'(t) = y'(t) = x_2(t)$$

and

$$\begin{aligned} x_2'(t) &= y''(t) \\ &= -6y(t) + 5y'(t) \\ &= -6x_1(t) + 5x_2(t) \end{aligned}$$

Writing this in matrix form, we have

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \mathbf{x}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example

Solving an Equation of the form $\mathbf{x}' = A\mathbf{x}$
Find the solution to the matrix ODE in the previous example.

Solution

$$A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ -6 & 5 - \lambda \end{pmatrix}$$

Find Eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= (-\lambda)(5 - \lambda) - (1)(-6) = 0 \\ \Rightarrow \lambda^2 - 5\lambda + 6 &= 0 \\ \Rightarrow \lambda_1 = 2, \qquad & \lambda_2 = 3 \end{aligned}$$

Find Eigenvectors:

$$\lambda_1 = 2 \qquad A - 2I = \begin{pmatrix} -2 & 1 \\ -6 & 5 - 2 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2v_1 + v_2 \\ -6v_1 + 3v_2 \end{pmatrix}$$

$$\Rightarrow v_2 = 2v_1 \qquad v^{(1)} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix}$$

Solving Nonhomogeneous Systems of Differential Equations

Variation of Parameters

The general solution to the system of first order differential equation

$$\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{g}}(t)$$

where $\mathbf{A}$ is a constant $n \times n$ matrix is given by

$$\vec{\mathbf{x}}(t) = \Psi(t) \cdot \left(\vec{c} + \int \Psi^{-1}(t)\vec{g}(t)\ dt \right)$$

or

$$\vec{\mathbf{x}}(t) = \Psi(t) \cdot \left(\Psi^{-1}(t_0)x(t_0) + \int_{t_0}^{t} \Psi^{-1}(t)\vec{g}(t)\ dt \right)$$

Laplace Transformations

The Laplace tranform of general solution to the system of first order differential equation

$$\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{g}}(t)$$

where $\mathbf{A}$ is a constant $n \times n$ matrix is given by

$$\vec{\mathbf{X}}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \left(\vec{\mathbf{G}}(s) + \vec{\mathbf{x}}(0) \right)$$

and the solution is given by

$$\vec{\mathbf{x}}(t) = \mathcal{L}^{-1} \left\{ (s\mathbf{I} - \mathbf{A})^{-1} \left(\vec{\mathbf{G}}(s) + \vec{\mathbf{x}}(0) \right) \right\}$$