

Sample Final Exam

Solutions

#1

part a

$$\frac{dz}{dt} = z^2 \sin(t)$$

→ linear, first-order, and separable

to solve...

$$\int \frac{dz}{z^2} = \int \sin(t) dt$$

$$\rightarrow -z^{-1} = -\cos(t) + C \rightarrow z^{-1} = \cos(t) + C$$

$$z(t) = \frac{1}{\cos(t) + C}$$

part b

$$y' = y \cdot \cos(x) + y - 1$$

→ linear, first order, solve via integrating factor

↳ rewrite as

$$\frac{dy}{dx} - y \cdot \cos(x) - y = 1$$

$$\frac{dy}{dx} - (\cos(x) + 1)y = 1$$

of the form $y' + p(x)y = g(x)$
 → solve via integrating factors

to solve:

$$y(x) = \frac{1}{\mu(x)} \left[\int g(x)\mu(x) dx + C \right], \quad \mu(x) = e^{\int p(x) dx}$$

$$\mu(x) = \exp \left[\int -(\cos(x) + 1) dx \right]$$

$$\mu(x) = \exp \left[-\sin(x) - x \right]$$

Leave in this form

$$\rightarrow y(x) = \frac{1}{e^{-(\sin(x)+x)}} \cdot \left[\int e^{(-\sin(x)-x)} dx + C \right]$$

#2

part a

$$y_1 = \sin(t) \quad y_2(t) = 1 + \sin(t)$$

check Wronskian: $W(y_1(t), y_2(t)) =$

$$\begin{vmatrix} \sin(t) & 1 + \sin(t) \\ \cos(t) & \cos(t) \end{vmatrix}$$

$$= \sin(t)\cos(t) - (1 + \sin(t))\cos(t)$$

$$= -\cos(t) \neq 0 \text{ unless } t = \frac{(2n+1)\pi}{2}$$

True

→ Since the Wronskian does not always equal to zero, the functions are linearly independent if $t \neq \frac{(2n+1)\pi}{2}$ where $n = 0, \pm 1, \pm 2, \dots$

part b

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

to check linear independence, check the determinant of the matrix with columns v_1 and v_2

$$\det \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} = -2 - 0 = -2 \neq 0$$

True

→ Since the determinant $\neq 0$, v_1 and v_2 are linearly independent

part c

$$y''' + y^3 = \cos(x)$$

→ this is third order, but the ODE is nonlinear since there is the "y³" term

final answer: False

#3

part a $y'' + y = xe^{-x} + \cos(x)$

In order to guess the form of the particular solution,
we need to know the homogeneous solution.

$$y'' + y = 0 \rightarrow \lambda^2 + 1 = 0 \quad \lambda = \pm i$$

$$\rightarrow \boxed{y_h = c_1 \cos(x) + c_2 \sin(x)}$$

Since the non-homogeneous term is
 $g(x) = \underbrace{xe^{-x}}_{\curvearrowleft} + \underbrace{\cos(x)}_{\curvearrowright}$ → sine/cosine type
 ↴ exponential-polynomial type

$$\rightarrow y_p(x) = (Ax + B)e^{-x} + \underbrace{C\cos(x) + D\sin(x)}_{\text{this portion is the homog. sol. so multiply by } x.}$$

$$\rightarrow \boxed{y_p(x) = (Ax + B)e^{-x} + Cx\cos(x) + Dx\sin(x)}$$

part b $y'' + y = \sin(7x) + (x^4 - 3x^2 + 2x - 1)$

→ same homogeneous problem as before...
 no repeated terms.

$$\boxed{y_p = A\sin(7x) + B\cos(7x) + Cx^4 + Dx^3 + Ex^2 + Fx + G}$$

#4

$$x^2 y'' - x(x+4)y' + (2x+6)y = x^4 e^x$$

part a

$y = x^2, y' = 2x, y'' = 2 \rightarrow$ plug in to homog.
problem

$$2x^2 - 2x^2(x+4) + (2x+6)x^2 = 0$$

$$\cancel{2x^2} - \cancel{2x^3} - 8x^2 + \cancel{2x^3} + 6x^2 = 0 \quad \checkmark$$

part b

first find homogeneous solution: (need 2nd fund. sol.)

$$y'' - \frac{x+4}{x}y' + \frac{2x+6}{x^2}y = x^2 e^x$$

use reduction of order!

→ 2 methods:

① assume $y_2(x) = u(x)y_1(x)$

② Abel's thm.

$$\text{Abel's thm} \rightarrow y_2 = y_1 \int \frac{w}{y_1^2} dx$$

$$\rightarrow y_2 = x^2 \cdot \int \frac{e^{-\int -\frac{x+4}{x} dx}}{x^4} dx$$

$$= x^2 \cdot \int \frac{e^{\int \frac{x+4}{x} dx}}{x^4} dx$$

$$= x^2 \int \frac{x^4 e^x}{x^4} dx = \boxed{x^2 e^x}$$

$$w = e^{-\int p(x) dx}$$

$$e^{\int 1 + \frac{4}{x} dx} = e^x e^{\ln(x^4)} = x^4 e^x$$

Thus, $y_1 = x^2$, $y_2 = x^2 e^x$, and

$$\underline{y_h = c_1 x^2 + c_2 x^2 e^x}$$

Now, for the particular solution!

Variation of parameters is what we should use!

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$\text{where } u_1(x) = - \int \frac{g(x)y_2(x)}{W} dx$$

$$u_2(x) = \int \frac{g(x)y_1(x)}{W} dx$$

$$\rightarrow u_1(x) = - \int \frac{x^2 e^x \cdot x^2}{x^4 e^x} dx = -e^x$$

$$\rightarrow u_2(x) = \int \frac{x^2 e^x \cdot x^2}{x^4 e^x} dx = \frac{1}{3}x^3$$

$$\rightarrow y_p(x) = -e^x x^2 + x^2 e^x \cdot \left(\frac{1}{3}x^3\right)$$

∴ $\boxed{y(x) = c_1 x^2 + c_2 x^2 e^x - x^2 e^x + \frac{1}{3}x^5 e^x}$

5

part a

$$y'' + 4y' + 4y = 0$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda + 2)^2 = 0 \rightarrow \lambda = -2, \lambda = -2$$

$$\rightarrow y = c_1 e^{-2t} + c_2 t e^{-2t}$$

$$y(0) = 1 \rightarrow c_1 = 1$$

$$y'(0) = 0 \rightarrow -2 + c_2 = 0 \rightarrow c_2 = 2$$

$$y = e^{-2t} + 2t e^{-2t}$$

part b

$$\text{let } u_1 = y \quad \left. \begin{array}{l} \\ u_2 = y' \end{array} \right\} \rightarrow \begin{array}{l} u'_1 = y' = u_2 \\ u'_2 = y'' = -4u_1 - 4u_2 \end{array}$$

$$\rightarrow \boxed{\begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \hat{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

part c

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \quad A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -4 & -4-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\boxed{\lambda^2 + 4\lambda + 4 = 0}$$

$$\boxed{\lambda = -2}$$

→ repeated e-values

$$\lambda_1 = -2$$

$$(A - \lambda I) \hat{z}_1 = 0$$

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} \hat{z}_{11} \\ \hat{z}_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} 2\hat{z}_{11} + \hat{z}_{12} = 0 \\ \hat{z}_{11} = 1, \hat{z}_{12} = -2 \end{array}$$

$$\rightarrow \hat{z}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\lambda_2 = -2$$

"eigenvector" $\hat{\eta}$

$$(A - \lambda I) \hat{\eta} = \hat{z}_1$$

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} \eta_{11} \\ \eta_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\left. \begin{array}{l} 2\eta_{11} + \eta_{12} = 1 \\ -4\eta_{11} - 2\eta_{12} = -2 \end{array} \right\} \rightarrow \eta_{11} = 1, \eta_{12} = -1$$

$$\hat{\eta}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Fundamental Sols

$$\hat{x}_1 = \hat{z}_1 e^{-2t}$$

$$\hat{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}$$

$$\begin{array}{l} \hat{x}_2 = \hat{z}_1 t e^{-2t} + \hat{\eta}_1 e^{-2t} \\ \hat{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} t e^{-2t} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} \end{array}$$

General Sol

$$\hat{x} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} t e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

Solving for $c_1 + c_2$ using the initial conditions, we have...

$$\hat{x}(t) = \begin{bmatrix} c_1 e^{-2t} + c_2 (te^{-2t} + e^{-2t}) \\ c_1(-2)e^{-2t} + c_2 (-2te^{-2t} - e^{-2t}) \end{bmatrix}$$

$$\begin{array}{l} c_1 + c_2 = 1 \\ -2c_1 - c_2 = 0 \end{array} \quad \begin{array}{l} c_2 = -2c_1 \\ -c_1 = 1 \end{array} \Rightarrow c_1 = -1, c_2 = 2$$

$$\hat{x}(t) = \begin{bmatrix} e^{-2t} + 2te^{-2t} \\ -4te^{-2t} \end{bmatrix}$$

since $y(t) = x_1(t)$, we have $y(t) = e^{-2t} + 2te^{-2t}$
 also, note that since $x_2(t) = y'$, we get the derivative
 of y in the second entry of the vector solution.

#6

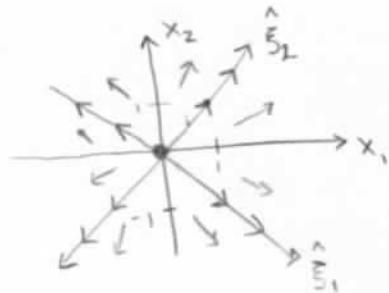
$$\left. \begin{array}{l} x_1 = y, \\ x_2 = y' \\ x_3 = y'' \\ x_4 = y''' \end{array} \right\} \rightarrow \begin{array}{l} x'_1 = y' = x_2 \\ x'_2 = y'' = x_3 \\ x'_3 = y''' = x_4 \\ x'_4 = y'''' = -4y''' + 3y' - y \\ = -4x_4 + 2x_1 \end{array} \quad \hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\rightarrow \hat{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & -4 \end{bmatrix} \hat{x}, \quad \hat{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

#7

part a

$$\lambda_1 = \frac{1}{2}, \hat{\xi}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \lambda_2 = 2, \hat{\xi}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



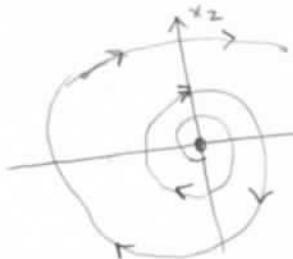
→ since both $\lambda_1 + \lambda_2$ are > 0 , the origin $(0,0)$ is unstable

part b

$$\lambda_1 = 1+i, \hat{\xi}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\begin{aligned} \rightarrow \hat{x}_1 &= \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(1+i)t} = \begin{bmatrix} -i(e^t)(\cos(t) + i\sin(t)) \\ e^t(\cos(t) + i\sin(t)) \end{bmatrix} \\ &= e^t \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + ie^t \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix} \end{aligned}$$

real part imag. part

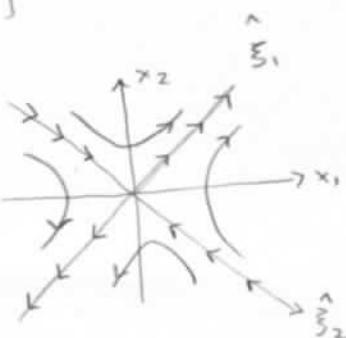


unstable spiral since $\operatorname{re}(\lambda) > 0$ (e^t -term) and the parametric eqns yield clockwise motion.

part c

$$\lambda_1 = \sqrt{122}, \hat{\xi}_1 = \begin{bmatrix} 11 \\ 1+\sqrt{122} \end{bmatrix}$$

$$\lambda_2 = -\sqrt{122}, \hat{\xi}_2 = \begin{bmatrix} 11 \\ 1-\sqrt{122} \end{bmatrix}$$



unstable saddle

#8 from earlier, we know the e-vols/vectors
 homog sol $\hat{x}_n(t) = c_1 \begin{bmatrix} 1 \\ 1+\sqrt{122} \end{bmatrix} e^{\sqrt{122}t} + c_2 \begin{bmatrix} 1 \\ 1-\sqrt{122} \end{bmatrix} e^{-\sqrt{122}t}$

neither of the funda. sols are repeated through the forcing term, so we can use undetermined coeffs, Laplace transforms, or variation of params in a straight forward fashion.

Variation of params vector of constants

$$* \hat{x}(t) = \underbrace{\Phi(t) \hat{c}}_{\hookrightarrow \text{fundamental matrix}} + \underbrace{\Phi(t)}_{\text{fundamental matrix}} \left[\Phi^{-1}(t) g(t) dt \right]$$

$$\Rightarrow \Phi(t) = \begin{bmatrix} 1 & e^{\sqrt{122}t} & -1 & e^{-\sqrt{122}t} \\ 1 & (1+\sqrt{122})e^{\sqrt{122}t} & (1-\sqrt{122})e^{\sqrt{122}t} & (1-\sqrt{122})e^{-\sqrt{122}t} \end{bmatrix}$$

$$\Phi^{-1}(t) = \frac{1}{-22\sqrt{122}} \begin{bmatrix} (1-\sqrt{122})e^{-\sqrt{122}t} & -1 & e^{-\sqrt{122}t} \\ -(1+\sqrt{122})e^{-\sqrt{122}t} & (1) & e^{\sqrt{122}t} \end{bmatrix}$$

$$g(t) = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix} \rightarrow \text{plug into } (*) \text{ and solve...}$$

#9

a. $F(s) = \frac{s}{s^2 + 1}$

b. $G(s) = \frac{-\pi/2s}{e^{-s^2} + 1}$

c. $\frac{2!}{(s-5)^3} = H(s)$

d. $G(s) = \sin\left(\frac{\pi}{4}\right)e^{-\pi/4s}$

#10

a. $f(t) = \frac{1}{2} \sin(2t)$

b. $G(s) = \frac{2}{(s-1)^2 + 1} = 2e^t \sin(t)$

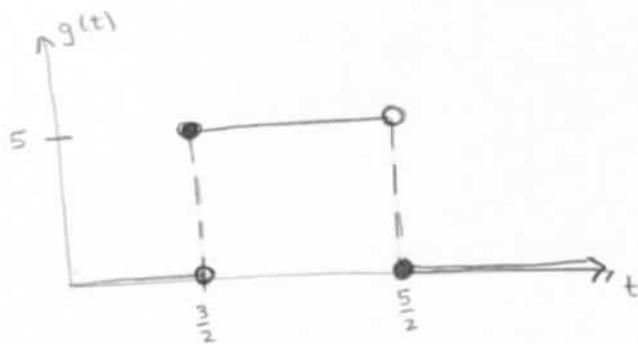
↑ note: complex roots
in denominator!
use rule #14

c. $H(s) = e^{-2s} \left[\frac{-1}{s-4} + \frac{1}{s+3} \right]$
 $h(t) = u_2(t) \cdot \begin{bmatrix} -e^{4(t-2)} & -e^{-3(t-2)} \\ -e^{4(t-2)} & +e^{-3(t-2)} \end{bmatrix}$

d. $g(t) = \int_0^t f(\tau) \cdot g(t-\tau) d\tau$

#11

part a



part b

$$(s^2 + \frac{1}{4}s + 1) Y(s) = \frac{5}{s} (e^{-\frac{3}{2}s} - e^{-\frac{5}{2}s})$$

$$Y(s) = (e^{-\frac{3}{2}s} - e^{-\frac{5}{2}s}) \cdot \underbrace{\frac{5}{s} \cdot \frac{1}{s^2 + \frac{1}{4}s + 1}}_{H(s)}$$

$$H(s) = \frac{s}{s(s^2 + \frac{1}{4}s + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + \frac{1}{4}s + 1}$$

$$A = 5 \quad B = -5, \quad C = -\frac{5}{4}$$

$$H(s) = \frac{s}{s} - \frac{\frac{5s}{s} + \frac{5}{4}}{(s + \frac{1}{8})^2 + \frac{63}{64}} = \frac{5}{s} - \frac{\frac{5(s + \frac{1}{8})}{(s + \frac{1}{8})^2 + \frac{63}{64}}}{(s + \frac{1}{8})^2 + \frac{63}{64}}$$

$$\rightarrow f(t) = 5 - 5 \cos(\sqrt{\frac{63}{64}}t) e^{-\frac{1}{8}t} - \frac{5}{8} \cdot \sqrt{\frac{64}{63}} \sin(\sqrt{\frac{63}{64}}t) e^{-\frac{1}{8}t}$$

final ans:

$$y(t) = u_{\frac{3}{2}}(t) \cdot h(t - \frac{3}{2}) - u_{\frac{5}{2}}(t) h(t - \frac{5}{2})$$

#12

$$y'' + xy' + 2y = 0$$

$$\left. \begin{array}{l} y = \sum_{n=0}^{\infty} a_n x^n \\ y' = \sum_{n=0}^{\infty} a_{n+1} \cdot (n+1) x^n \\ y'' = \sum_{n=0}^{\infty} a_{n+2} \cdot (n+1)(n+2) x^n \end{array} \right\} \rightarrow \sum_{n=0}^{\infty} a_{n+2} (n+1)(n+2) x^n + \sum_{n=0}^{\infty} a_{n+1} (n+1) x^{n+1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2} (n+1)(n+2) x^n + \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\rightarrow a_2(1) \cdot (2) + 2a_0 + \sum_{n=1}^{\infty} [(n+1)(n+2)a_{n+2} + (n+2)a_n] x^n = 0$$

$$\rightarrow 2a_0 + 2a_2 = 0$$

$$a_{n+2} = \frac{(n+2)a_n}{(n+1)(n+2)}$$

recurrence relation