MATH 234 (02)
SAMPLE FINAL EXAM

Please complete the following problems.

1. For the following first-order differential equations, write down which method you could use to solve them. Choose one of *linear*, *separable*, or *exact*. If you choose exact, you must show that the equation is exact. **Note: You do not have to solve the differential equations!**
   (a) $\frac{dz}{dt} = z^2 \sin(t)$
   (b) $y' = y\cos(x) + y - 1$

2. State whether the following statements are true or false, and explain why.
   (a) The functions $y_1 = \sin(t)$ and $y_2 = 1 + \sin(t)$ are linearly independent.
   (b) The vectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ are linearly independent.
   (c) The differential equation $y''' + y^3 = \cos(x)$ is a third order linear equation.

3. For the following equations, write down the form of the particular solution, using the method of undetermined coefficients. **You do not need to find the values of the coefficients!**
   (a) $y'' + y = xe^{-x} + \cos(x)$
   (b) $y'' + y = \sin(7x) + \left(x^4 - 3x^2 + 2x - 1\right)$

4. Consider the equation
$$x^2y'' - x(x+4)y' + (2x+6)y = x^4e^x$$
   (a) Check that $y_1 = x^2$ is a solution of the associated homogeneous problem.
   (b) Find the general solution of this equation.

5. Consider the initial-value problem
$$\begin{aligned} y'' + 4y' + 4y &= 0 \\ y(0) = 1, \quad y'(0) &= 0 \end{aligned}$$
   (a) Solve this initial value problem.
   (b) Write this initial value problem as a system of two first order differential equations.
   (c) Find the eigenvalues and eigenvector(s) for the system you found in part b). Describe the behavior of solutions in the phase plane.

6. Rewrite the following initial value problem as a system of first-order equations with an initial condition. Express your answer in matrix form. **Note that you do not need to solve this problem, just rewrite it!**
$$\begin{aligned} y'''' + 4y''' - 3y + y &= 0 \\ y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 2, \quad y'''(0) &= -1 \end{aligned}$$

7. Describe the phase plane behavior for each of the following linear systems and classify the origin as stable or unstable.
   (a)
$$\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x}$$

(b)

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x}$$

(c)

$$\mathbf{x}' = \begin{pmatrix} -1 & 11 \\ 11 & 1 \end{pmatrix} \mathbf{x}$$

8. Find the general solution to the differential equation

$$\mathbf{x}' = \begin{pmatrix} -1 & 11 \\ 11 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{t} \\ e^{-t} \end{pmatrix}$$

9. Find the Laplace Transform of the following functions.
   (a) $f(t) = \cos(t)$
   (b) $g(t) = u_{\frac{\pi}{2}}(t)\sin(t - \frac{\pi}{2})$
   (c) $h(t) = t^2 e^{5t}$
   (d) $g(t) = \delta\left(t - \dfrac{\pi}{4}\right)\sin(t)$

10. Find the Inverse Laplace Transform of the following functions.
   (a) $F(s) = \dfrac{1}{s^2+4}$
   (b) $G(s) = \dfrac{2}{s^2-2s+2}$
   (c) $H(s) = \dfrac{e^{-2s}}{s^2-s-12}$
   (d) $Q(s) = F(s)\cdot G(s)$

11. A certain mass-spring-damper system satisfies the initial value problem

$$y'' + \frac{1}{4}y' + y = g(t)$$
$$y(0) = 0, \quad y'(0) = 0$$

where $g(t) = 5u_{\frac{3}{2}}(t) - 5u_{\frac{5}{2}}(t)$.
   (a) Sketch a graph of $g(t)$.
   (b) Solve the initial value problem using the Laplace Transform.

12. Consider the differential equations

$$y'' + xy' + 2y = 0$$

   (a) Find the recurrence relationship if we assume that $y(x)$ can be represented as a power series about the point $x_0 = 0$.
   (b) Find the first four term in each of the two linearly independent solutions.