## Math 234 - Exam \#2 Study Guide

## Spring Quarter 2013

## Laplace Transforms

The Laplace Transform is a technique that we used to help us solve initial value problems of the form

$$
a y^{\prime \prime}+b y^{\prime}+c y=g(t), \quad y(0)=y_{0}, \quad y^{\prime}(0)=y_{0}^{\prime}
$$

specifically when the function $g(t)$ was not continuous or continuously differentiable. The Laplace transform can be used to solve a wider class of differential equations (higher order, some non-constant coefficient ODEs, etc). It can also be applied integral equations where the integral looks like a convolution!

## Definitions of the Laplace Transform

## Definition:

The Laplace transform of a function $f(t)$ is given by

$$
\mathcal{L}\{f(t)\}=\int_{0}^{\infty} e^{-s t} f(t) d t
$$

On the exam you will be given the same table of transforms that you were given in class.

Definition:
The Inverse Laplace transform of a function $F(s)$ is denoted by

$$
\mathcal{L}^{-1}\{F(s)\}=f(t)
$$

The inverse Laplace transform is typically found by using the method of partial fractions and then using the table of Laplace transforms.

Remember that there are various cases to partial fractions; some can be tricky to remember. The main goal is to look at the roots of the denominator.

Consider the following scenarios (where $P(s)$ is a polynomial with degree at least one smaller than the degree of the denominator).:
$G(s)=\frac{P(s)}{(s-a)(s-b)}$, where $a \neq b$
$G(s)=\frac{P(s)}{(s-a)^{2}}$
$G(s)=\frac{P(s)}{(s-a)^{n}}$
$G(s)=\frac{P(s)}{a s^{2}+b s+c}$,
where $a s^{2}+b s+c=0$ has complex roots
$G(s)=\frac{A}{s-a}+\frac{B}{s-b}$
$G(s)=\frac{A}{s-a}+\frac{B}{(s-a)^{2}}$
$G(s)=\frac{A_{1}}{s-a}+\frac{A_{2}}{(s-a)^{2}}+\ldots+\frac{A_{n}}{(s-a)^{n}}$
$G(s)=\frac{A s+B}{(s-\alpha)^{2}+\beta}$ from completing the square

## Example

Inverse Laplace Transform (via Partial Fractions)
Find the inverse transform of

$$
H(s)=\frac{1}{s^{2}+3 s}
$$

## Solution

First we start by using a partial fraction expansion on the function

$$
H(s)=\frac{1}{s^{2}+3 s}=\frac{A}{s}+\frac{B}{s+3}
$$

Solving for $A$ and $B$, we find

$$
H(s)=\frac{1 / 3}{s}-\frac{1 / 3}{s+3}
$$

Using the Laplace table (specifically Rule \#1 and Rule \#2), we find

$$
h(t)=\frac{1}{3}-\frac{1}{3} e^{-3 t}
$$

One of the tricky examples is find the inverse Laplace transform of a function that involves exponentials. Consider the following example:

## Example

Inverse Laplace Transform (via Partial Fractions)
Find the inverse transform of

$$
F(s)=\frac{e^{-s}-e^{-3 s}}{s^{2}+3 s}
$$

## Solution

Since the transform involves exponentials times some function of $s$, Rule $\# 13$ with be the important rule. It says the following:

$$
\mathcal{L}^{-1}\left\{e^{-c s} H(s)\right\}=u_{c}(t) h(t-c)
$$

where $h(t)=\mathcal{L}\{H(s)\}$. Therefore, let's rewrite $F(s)$ as

$$
F(s)=e^{-s} \frac{1}{s^{2}+3 s}-e^{-3 s} \frac{1}{s^{2}+3 s}=e^{-s} H(s)-e^{-3 s} H(s)
$$

where

$$
H(s)=\frac{1}{s^{2}+3 s}
$$

Thus, Rule $\# 13$ says that $f(t)=u_{1}(t) h(t-1)-u_{3}(t) h(t-3)$ where $h(t)$ was found in the previous example. Thus, the final answer is given by

$$
f(t)=u_{1}(t)\left(\frac{1}{3}-\frac{1}{3} e^{-3(t-1)}\right)-u_{3}(t)\left(\frac{1}{3}-\frac{1}{3} e^{-3(t-3)}\right)
$$

## Using the Laplace Transform to Solve ODEs

In order to use Laplace Transforms to solve differential equations, we need to discuss what happens if we take the Laplace transform of the derivative of a function. We proved the following theorem in class using integration by parts.

## Theorem

Assume that $f(t)$ is an n-times differentiable function whose Laplace transform is given by $F(s)$. Then: The Laplace transform of $f^{\prime}(t)$ is given by

$$
\mathcal{L}\left\{f^{\prime}(t)\right\}=s F(s)-f(0)
$$

The Laplace transform of $f^{\prime \prime}(t)$ is given by

$$
\mathcal{L}\left\{f^{\prime \prime}(t)\right\}=s^{2} F(s)-s f(0)-f^{\prime}(0)
$$

The Laplace transform of $f^{(n)}(t)$ is given by

$$
\mathcal{L}\left\{f^{(n)}(t)\right\}=s^{n} F(s)-s^{n-1} f(0)-\ldots-s f^{(n-2)}(0)-f^{(n-1)}(0)
$$

To solve initial value problems such as

$$
a y^{\prime \prime}+b y^{\prime}+c y=g(t), \quad y(0)=y_{0}, y^{\prime}(0)=y_{0}^{\prime}
$$

using Laplace transforms, we followed these steps:

1. Take the Laplace transform of both sides of the differential equation using $Y(s)$ to represent the Laplace transform of the unknown function $y(t)$.
2. Solve the algebraic equation from Step 1 for the function $Y(s)$. This gives us the Laplace transform of our solution.
3. Find the inverse Laplace transform of $Y(s)$ from Step 2. This gives the solution $y(t)=\mathcal{L}^{-1}\{Y(s)\}$.

One of the advantages of using Laplace transforms is that the initial conditions are automatically satisfied. Another is that we can deal with discontinuous forcing functions. So, what are some of those discontinuous functions that we dealt with?

## DEfinition:

The Unit Step Function: The unit step function $u_{c}(t)$ is defined as the piecewise function

$$
u_{c}(t)= \begin{cases}0 & \text { if } t<c \\ 1 & \text { if } t \geq c\end{cases}
$$

The graph of $u_{c}(t)$ is given in the figure below.


One of the nice things about the unit-step function is that it allows us to quickly write down piecewise functions on a single line!

## Example

Write the piecewise function

$$
u(x)= \begin{cases}0 & 0 \leq t<1 \\ 1-(t-2)^{2} & 1 \leq t<3 \\ 0 & t>3\end{cases}
$$

in unit-step notation.

## Solution

This is pretty straight-forward. For each piece $f_{j}(t)$ that starts at $t=a_{j}$ and ends at $t=b_{j}$, we simply write the component as $f_{j}(t)\left(u_{a_{j}}(t)-u_{b_{j}}(t)\right)$ since the function is activated at $t=a_{j}$ and deactivated at $t=b_{j}$. We then add the results together. Thus, for this problem, we have

$$
f(t)=0\left(u_{0}(t)-u_{1}(t)\right)+\left(1-(t-2)^{2}\right)\left(u_{1}(t)-u_{3}(t)\right)+0\left(u_{3}(t)\right)
$$

and finally,

$$
f(t)=\left(1-(t-2)^{2}\right)\left(u_{1}(t)-u_{3}(t)\right)
$$

## Definition:

The Impulse Function: The impulse function $\delta(t)$ is defined by the following properties:

$$
\delta(t-c)=0 \text { if } t \neq c \int_{-\infty}^{\infty} \delta(t) d t=1
$$

Note that

$$
\int_{-\infty}^{\infty} \delta(t-c) f(t) d t=f(c), \quad \int_{0}^{\infty} \delta(t-c) f(t) d t=f(c), c>0
$$

In theory, we can imagine that the graph of $\delta(t-c)$ would look like the one given in the figure below:


## The Convolution

Finally, we introduced the convolution to assist with inverse Laplace transforms.

## Definition:

The convolution of two functions $f(t)$ and $g(t)$ is given by

$$
f * g=\int_{0}^{t} f(t-\tau) g(\tau) d \tau
$$

There are several important perperties of the convolution:

1. $f * g=g * f$ (commutative property)
2. $f *\left(g_{1}+g_{2}\right)=f * g_{1}+f * g_{2}$ (distributive property)
3. $f *(g * h)=(f * g) * h$ (associative property)

Theorem
The Inverse Transform of the Product of Two Transforms: If $\mathcal{L}\{f(t)\}=F(s)$ and $\mathcal{L}\{g(t)\}=G(s)$ and $H(s)=F(s) \cdot G(s)$, then the inverse transform of $H(s)$ is given by

$$
h(t)=\mathcal{L}^{-1}\{H(s)\}=f(t) * g(t)
$$

We can use the convolution to assist with the inverse Transform when we can identify the inverse transforms of each term in a product. Consider the following example:

## Example

Finding the Inverse Laplace Transform (via the Convolution Theorem)
Find the inverse transform of

$$
H(s)=\frac{1}{s^{2}+3 s}
$$

## Solution

First we start by thinking of $H(s)$ as the product of two transforms $F(s)$ and $G(s)$ where

$$
H(s)=\frac{1}{s} \cdot \frac{1}{s+3}=F(s) \cdot G(s)
$$

By using the table, we see that $f(t)=1$ and $g(t)=e^{-3 t}$. Using the convolution theorem, we find that

$$
\begin{aligned}
h(t) & =f * g \\
& =\int_{0}^{t} 1 \cdot e^{-3 \tau} d \tau \\
& =-\frac{1}{3} e^{-3 \tau} \|_{0}^{t} \\
& =-\frac{1}{3}\left(e^{-3 t}-1\right)
\end{aligned}
$$

Notice that we find the same answer using the convolution as we did using partial fractions.

## SERIES Solutions

Now, we consider solving non-constant coefficient second order linear differential equation of the form:

$$
P(x) y^{\prime \prime}+Q(x) y^{\prime}+R(x) y=G(x)
$$

If we write this ode in standard form, we have

$$
y^{\prime \prime}+p(x) y^{\prime}+q(x) y=g(x)
$$

where

$$
p(x)=\frac{Q(x)}{P(x)}, \quad q(x)=\frac{R(x)}{P(x)}, \quad g(x)=\frac{G(x)}{P(x)} .
$$

In general, we have no method to solve this type of differential equation. So, instead, we assume that the solution of the differential equation has some series representation. By assuming a form of the solution, we can verify if it is correct by plugging it into the differential equation. However, the form that the solution takes varies depending on where we expand our solution.

## Regular (or Ordinary) Points

The easiest series solution method is when we find the solution in a neighborhood of a regular point (or ordinary point).

## Definition:

Consider the general second order differential equation

$$
P(x) y^{\prime \prime}+Q(x) y^{\prime}+R(x) y=0
$$

where $P(x), Q(x)$, and $R(x)$ are all polynomials, and there are no common factors in $P(x), Q(x)$, and $R(x)$. A point $x=x_{0}$ is a regular (or ordinary) point of the differential equation if $P\left(x_{0}\right) \neq 0$.

If we look for the series solution in the neighborhood of a regular point, then the series solution will take the form

$$
y(x)=\sum_{n=0}^{\infty} a_{n}\left(x-x_{0}\right)^{n}
$$

where we solve for the $a_{n}$ 's by

1. Plugging the series into the differential equation.
2. Re-indexing in order to equate like power of $x$ (or $x-x_{0}$ ).
3. Find the recursion relationship.
4. Write the first few terms in the series expansion.

Furthermore, we were also able to find both fundamental solutions, as well as the radius for which the series solution converged using the following theorem.

## Theorem

If $x_{0}$ is an ordinary point of the differential equation

$$
\begin{equation*}
P(x) y^{\prime \prime}+Q(x) y^{\prime}+R(x) y=0 \tag{1}
\end{equation*}
$$

that is, $p(x)=Q(x) / P(x)$ and $q(x)=R(x) / P(x)$ are analytic at $x=x_{0}$, then the general solution of (1) is given by

$$
y(x)=\sum_{n=0}^{\infty} a_{n}\left(x-x_{0}\right)^{n}=a_{0} y_{1}+a_{1} y_{2}
$$

where $a_{0}$ and $a_{1}$ are arbitrary constants, and $y_{1}$ and $y_{2}$ are linearly independent series solutions. Furthermore, $y(x)$ is analytic at $x=x_{0}$, and the radius of convergence for $y(x)$ is at least as large as the radius of convergence for the series expansion of $p(x)$ and $q(x)$.

In order to find the radius of convergence, we simply look for the distance between $x_{0}$ and the nearest singularity of $p(x)$ and $q(x)$.

## Example

Find the radius of convergence for the Taylor series of $f(x)$ about $x_{0}=2$ where $f(x)$ is given by

$$
f(x)=\frac{7 x-1}{x^{3}+x}
$$

## Solution

The singularities of $f(x)$ are when the denominator is zero. Solving for when $x^{3}+x=0$, we find

$$
x\left(x^{2}+1\right)=0 \quad \longrightarrow x=0, x= \pm i
$$

So, we consider the following cases:
$x=0$ : The distance from $x_{0}=2$ to $x=0$ is

$$
\rho=\sqrt{(2-0)^{2}+(0-0)^{2}}=2
$$

$x=i$ : The distance from $x_{0}=2$ to $x=+i$ is

$$
\rho=\sqrt{(2-0)^{2}+(0-1)^{2}}=\sqrt{5}
$$

$x=-i$ : The distance from $x_{0}=2$ to $x=-i$ is

$$
\rho=\sqrt{(2-0)^{2}+(0-(-1))^{2}}=\sqrt{5}
$$

Since the smallest radius is $\rho=2$, the series will converge for all $x$ that satisfies $|x-2|<2$ or specifically,

$$
\text { The radius of convergence is } \rho=2 \text { and the series will converge for all } x \text { such that } 0<x<4 \text {. }
$$

## Regular Singular Points

Definition:
Consider the general second order differential equation

$$
P(x) y^{\prime \prime}+Q(x) y^{\prime}+R(x) y=0,
$$

where $P(x), Q(x)$, and $R(x)$ are all polynomials, and there are no common factors in $P(x), Q(x)$, and $R(x)$. A point $x_{0}$ is a regular singular point of the differential equation if all three of the following conditions are satisfied:

1. $P\left(x_{0}\right)=0$
2. $\lim _{x \rightarrow x_{0}}\left(x-x_{0}\right) \frac{Q(x)}{P(x)}$ exists and is finite.
3. $\lim _{x \rightarrow x_{0}}\left(x-x_{0}\right)^{2} \frac{R(x)}{P(x)}$ exists and is finite.

If we look for the series solution in the neighborhood of a regular singular point, then the series solution will take the form

$$
y(x)=\left(x-x_{0}\right)^{r} \sum_{n=0}^{\infty} a_{n}\left(x-x_{0}\right)^{n}
$$

where we solve for $r$ and the $a_{n}$ 's by

1. Plugging the series into the differential equation.
2. Re-indexing in order to equate like power of $x$ (or $x-x_{0}$ ).
3. Evaluate the solution at the lowest power of $x$ (or $x-x_{0}$ ) in order to solve for $r$. Typically, we find two distinct $r$ values that are not integers apart.
4. Find the recursion relationship for each value of $r$.
5. Write the first few terms in the series expansion for each value of $r$. Each sum (for each of your $r$ values) corresponds to one fundamental solution.
6. Write the general solution as $y(x)=c_{1} y_{1}+c_{2} y_{2}$.
