

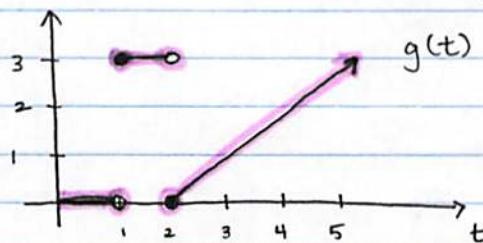
#1 a) $g(t) = 3u_1(t) - u_2(t)(5-t)$

I method:

Rewrite: $g(t) = 3u_1(t) - 3u_2(t) - (2-t)u_2(t)$

$$= 3(u_1(t) - u_2(t)) + (t-2)u_2(t)$$

turn "3" on @ $t=1$
turn "3" off @ $t=2$



alt: you could
create a table
of values

$$\mathcal{L}\{g(t)\} = \lim_{A \rightarrow \infty} \int_0^A e^{-st} [3u_1(t) - u_2(t)(5-t)] dt$$

$$= \lim_{A \rightarrow \infty} \left[\int_0^A e^{-st} 3u_1(t) dt - \int_0^A e^{-st} u_2(t)(5-t) dt \right]$$

$$= \lim_{A \rightarrow \infty} \left[\int_1^A e^{-st} 3 dt - \int_2^A e^{-st} (5-t) dt \right]$$

$$= \lim_{A \rightarrow \infty} \left[-\frac{3}{s} e^{-st} \Big|_1^A - 5 \int_2^A e^{-st} dt + \int_2^A e^{-st} t dt \right]$$

integrate by
parts

$$= \lim_{A \rightarrow \infty} \left[-\frac{3}{s} e^{-st} \Big|_1^A + \frac{5}{s} e^{-st} \Big|_2^A + \left(\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2} \right) \Big|_2^A \right]$$

$$= \lim_{A \rightarrow \infty} \left[-\frac{3}{s} e^{-sA} + \frac{5}{s} e^{-sA} - \frac{Ae^{-sA}}{s} - \frac{e^{-sA}}{s^2} + \frac{3}{s} e^{-s} - \frac{5}{s} e^{-2s} + \frac{2e^{-2s}}{s} + \frac{e^{-2s}}{s^2} \right]$$

$$G(s) = \frac{3}{s} e^{-s} - \frac{3}{s} e^{-2s} + \frac{e^{-2s}}{s^2}, \quad s > 0$$

b) $\mathcal{L}^{-1} \left\{ \frac{1}{s^3 + 2s^2 + s} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{s^2 + 2s + 1} \right\}$

$$= \mathcal{L}^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{(s+1)^2} \right\}$$

there are many methods, such as partial fractions, but I'll choose convolution!

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{(s+1)^2} \right\} &= \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} * \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2} \right\} \\ &= 1 * t e^{-t} \\ &= \int_0^t 1 \cdot \tau e^{-\tau} d\tau \\ &= -\tau e^{-\tau} - e^{-\tau} \Big|_0^t \\ &= -te^{-t} - e^{-t} + 1 \end{aligned}$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{(s+1)^2} \right\} = 1 - e^{-t} - te^{-t}$$

$$\#2 \quad y'' - 4y' + 13y = e^t * \delta(t-4) \quad y(0)=6, \quad y'(0)=12$$

take L.T.

$$s^2 Y(s) - s \cdot 6 - 12 - 4(sY(s) - 6) + 13Y(s) = \mathcal{L}\{e^t * \delta(t-4)\}$$

$$(s^2 - 4s + 13)Y(s) - 6s + 12 = \mathcal{L}\{e^t\} \mathcal{L}\{\delta(t-4)\}$$

$$(s^2 - 4s + 13)Y(s) - 6(s-2) = \frac{1}{s-1} \cdot e^{-4s}$$

Solve for $Y(s)$

$$Y(s) = \frac{6(s-2)}{s^2 - 4s + 13} + \frac{e^{-4s}}{s^2 - 4s + 13} \cdot \frac{1}{s-1}$$

*

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Find inverse transform:

$$(*) \mathcal{L}^{-1}\left\{\frac{6(s-2)}{s^2 - 4s + 13}\right\} : \quad \frac{6(s-2)}{s^2 - 4s + 4 + 9} = \frac{6(s-2)}{(s-2)^2 + 3^2}$$

↑

complex roots,

complete the square.

$$\mathcal{L}^{-1}\left\{\frac{6(s-2)}{(s-2)^2 + 3^2}\right\} = 6e^{2t} \cos(3t) \quad (\text{via } \#10)$$

$$(*) \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{((s-2)^2 + 3^2)(s-1)}\right\} : \quad \begin{aligned} &\text{ignore exponential for now...} \\ &\rightarrow \#13: u_4(t) \cdot h(t-4) \end{aligned}$$

$$H(s) = \frac{1}{((s-2)^2 + 3^2)(s-1)} = \frac{As + B}{(s-2)^2 + 3^2} + \frac{C}{s-1}$$

$$As(s-1) + B(s-1) + C[s^2 - 4s + 13] = 1$$

$$\text{plug in } s=1 \rightarrow 10C = 1 \rightarrow C = \frac{1}{10}$$

$$s=0 \rightarrow -B + \frac{1}{10}(13) = 1 \rightarrow B = \frac{3}{10}$$

$$s=2 \rightarrow 2A + \frac{3}{10}(1) + \frac{1}{10}(1) = 1$$

$$2A = -\frac{2}{10} \rightarrow A = -\frac{1}{10}$$

$$\therefore H(s) = \frac{1}{10} \left[\frac{-s+3}{(s-2)^2 + 3^2} + \frac{1}{s-1} \right]$$

$$= \frac{1}{10} \left[\frac{\underline{-(s-2+2)+3}}{(s-2)^2 + 3^2} + \frac{1}{s-1} \right]$$

$$= \frac{1}{10} \left[\frac{\underline{-(s-2)}+1}{(s-2)^2 + 3^2} + \frac{1}{s-1} \right]$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\}$$

$$h(t) = \frac{1}{10} \left[-e^{2t} \cos(3t) + \frac{1}{3} e^{2t} \sin(3t) + e^t \right]$$

$$\therefore y(t) = 6e^{2t} \cos(3t) + u_4(t) \cdot h(t-4)$$

$$y(t) = 6e^{2t} \cos(3t) + u_4(t) \frac{1}{10} \left[-e^{2(t-4)} \cos(3(t-4)) + \frac{1}{3} e^{2(t-4)} \sin(3(t-4)) + e^{t-4} \right]$$

#3 b) Since $x=1$ is a regular point of the ODE, let

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

In the series HW, we had to options for this ^{type of} problem.
This time, we use option #2.

$$\text{Let } z = x-1 \quad \rightarrow \quad x = z+1 \quad \frac{d}{dz} = \frac{d}{dx}$$

Thus the ODE becomes

$$(3 - (z+1)) y''(z) + y'(z) + ((z+1)^2 - 1) y(z) = 0$$

or

$$(2-z)y'' + y' + (z^2 + 2z)y = 0$$

and we seek a series solution of the form

$$y(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{since } x_0 = 1 \rightarrow z_0 = 0$$

plug in:

$$\begin{aligned} & \sum_{n=2}^{\infty} 2a_n n(n-1) z^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1) z^{n-1} + \sum_{n=1}^{\infty} a_n n z^{n-1} \\ & + \sum_{n=0}^{\infty} a_n z^{n+2} + \sum_{n=0}^{\infty} 2a_n z^{n+1} = 0 \end{aligned}$$

reindex:

$$\begin{aligned} & \sum_{n=0}^{\infty} 2a_{n+2} (n+2)(n+1) z^n - \sum_{n=1}^{\infty} a_{n+1} (n+1)(n) z^n + \sum_{n=0}^{\infty} a_{n+1} (n+1) z^n \\ & + \sum_{n=2}^{\infty} a_{n-2} z^n + \sum_{n=1}^{\infty} 2a_{n-1} z^n = 0 \end{aligned}$$

group :

$$2a_2(2)(1)z^0 + 2a_3(3)(2)z^1 - a_2(2)(1)z^1 + a_1(1)z^0 + a_2(2)z^1 + 2a_0z^0 \\ + \sum_{n=2}^{\infty} z^n \cdot \left[2a_{n+2}(n+2)(n+1) - a_{n+1}(n+1)(n) + a_{n+1}(n+1) + a_{n-2} + 2a_{n-1} \right] \\ \text{group!} = 0$$

collect like powers of z :

$$z^0: 4a_2 + a_1 = 0 \quad a_2 = -\frac{1}{4}a_1$$

$$z^1: 12a_3 - 2a_2 + 2a_2 + 2a_0 = 0 \quad a_3 = -\frac{1}{6}a_0$$

$$z^n: \sum_{n \geq 2} 2a_{n+2}(n+2)(n+1) - a_{n+1}(n+1)(n-1) + a_{n-2} + 2a_{n-1} = 0$$

$$a_{n+2} = \frac{a_{n+1}(n^2-1) - a_{n-2} - 2a_{n-1}}{2(n+2)(n+1)}$$

recurrence relationship

general solution:

$$y(z) = a_0 + a_1 z + (-\frac{1}{4}a_1)z^2 + (-\frac{1}{6}a_0)z^3 + \dots$$

higher order terms

$$y(z) = a_0(1 - \frac{1}{6}z^3 + \dots) + a_1(z - \frac{1}{4}z^2 + \dots)$$

fundamental solutions

Thus, fundamental solution are ...

$$y_1(z) = 1 - \frac{1}{6}z^3 + \dots$$

$$y_2(z) = z - \frac{1}{4}z^2 + \dots$$

in original variables

$$y_1(x) = 1 - \frac{1}{6}(x-1)^3 + \dots$$

$$y_2(x) = (x-1) - \frac{1}{4}(x-1)^2 + \dots$$

#4

a) False $f * g = g * f$

$$f * g = \int_0^t f(t-\tau) g(\tau) d\tau \quad \leftarrow \text{definition}$$

introduce a change of variables: $\alpha = t - \tau$ $d\alpha = -d\tau$
don't forget to change the limits!!!

if $\tau=0$, then $\alpha=t$
if $\tau=t$, then $\alpha=0$

Thus, the change of variables gives

$$f * g = \int_t^0 f(\alpha) g(t-\alpha) (-1) d\alpha$$

$$= \int_0^t f(\alpha) g(t-\alpha) d\alpha \quad \begin{matrix} \text{switch the order of} \\ \text{the limits} \rightarrow \text{neg sign change.} \end{matrix}$$

$$= \int_0^t g(t-\alpha) f(\alpha) d\alpha$$

$$= g * f \quad \checkmark$$

Faster proof:

$$f * g = \mathcal{L}^{-1}\{\mathcal{L}\{f * g\}\} = \mathcal{L}^{-1}\{F(s) \cdot G(s)\}$$

$$= \mathcal{L}^{-1}\{G(s) \cdot F(s)\}$$

$$= g(t) * f(t) \quad \checkmark$$

b) False. The correct format is

$$f(t) = \cos(t) \cdot \left[u_{\frac{\pi}{2}}(t) - u_{\frac{3\pi}{2}}(t) \right]$$

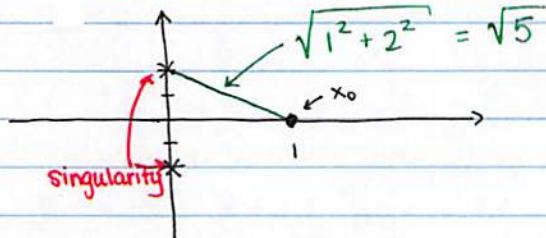
c) Writing the differential equation in standard form, we have

$$y'' - \frac{x}{x^2+4} y = 0$$

A lower bound on the radius of convergence is the radius of convergence for $p(x) = \frac{-x}{x^2+4}$

the series expansion about $x_0 = 1$ for

Using complex analysis, we have 2 singularities for $p(x)$: $x = +2i$ and $x = -2i$



True

a lower bound on the radius of convergence is given by $\sqrt{5}$