Math 2340 - Winter Quarter 2015

Exam #1 Study Guide

Preliminary Material

In the first few lectures, we covered some basic definitions and classifications of differential equations. Here are a few of the classifications:

Definition: Order of an ODE

The order of a differential equation is determined by the order of the highest derivative of the equation.

For example,

$$\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} \cdot \frac{d^3y}{dx^3} = \cos(x)$$

is a forth-order ODE because the highest derivative is a forth order derivative.

Definition: Linear vs. Nonlinear ODE

A differential equation is linear if the dependent variable, as well as all of it's derivatives, are linear. For example, a linear ODE can be written as

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_0(x)y = f(x)$$

This means that $y'' + 3x^3y' + \cos(x)y = e^x$ is linear since y and all of its derivatives show up linearly.

On the other hand, $y'y = 1$ is nonlinear because of the $y'y$ term.

First Order Differential Equations

In class, we discussed several ways to solve a first-order differential equation. The biggest step to solving an ODE is to classify the type. The most general first order ODE can be written in the form

$$y' = f(x, y).$$

Once you know that the ODE is first-order, you have the following "flow-chart" to help you determine how to solve the problem:

1. **Is the ODE separable?** Can the ODE be written in the form $y' = f(x)g(y)$? If so, separate and integrate. *Try solving the ODE for y' and then look to see if you can factor the right-hand side. Never spend more than a minute trying to check if the ODE is separable.*

2. **Is the ODE linear?** Can the ODE be written in the form $y' + p(x)y = g(x)$? If so, you can either solve via the linear integrating factor ($\mu(x)$), or using VOP.

3. **Is the ODE exact?** In other words, if written in the form $M(x, y) + N(x, y)y' = 0$, does $M_y = N_x$? If so, solve as an exact ODE.

4. **Is the ODE a Bernoulli type?** That is, can the ODE be written in the form $y' + p(x)y = g(x)y^n$? If so, make the substitution $v(x) = (y(x))^{1-n}$ and solve as a linear ODE.

If all of the above techniques fail, you can always analyze the behavior of solutions via a direction field. These techniques or approaches are summarized below.

Separable Differential Equations

A first order differential equation is *separable* if we can write it in the form

$$\frac{dy}{dx} = f(x) \cdot g(y) \quad \Longrightarrow \quad \int \frac{dy}{g(y)} = \int f(x)\ dx + c$$

In other words, if we can separate the x's and the y's, we can solve the equation by integrating each side of the equation.

Linear Differential Equations

There are two different techniques that we learned for solving first-order ODEs of the form

$$y' + p(x)y = g(x).$$

They are:

1. Linear Integrating Factors
2. Variation of Parameters

Both techniques are explored below.

Linear Integrating Factors

Definition: Linear Integrating Factor (Linear First Order ODE)

The integrating factor for a first order linear ODE of the form

$$y'(x) + p(x)y(x) = g(x)$$

is a function $\mu(x)$ such that when the above differential equation is multiplied by $\mu(x)$ yielding

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)g(x),$$

the left-hand side can be rewritten as a product rule of the form

$$\frac{d}{dx}\left(\mu(x)\, y(x)\right) = \mu(x)g(x).$$

The unknown function $y(x)$ can be solved for by integrating both sides of the equation with respect to x.

Variation of Parameters (VoP)

Definition: Variation of Parameters (Linear First Order ODE)

Consider the first-order, linear ODE of the form

$$y'(x) + p(x)y(x) = g(x).$$

You can find the solution by separating this into two separate problems:

1. Solve the homogeneous problem $y'(x) + p(x)y(x) = 0$. This ODE is separable, so you can separate and integrate. Call this solution $y_h(x)$.

2. The solution $y_h(x)$ will depend on a constant of integration c. Rename this solution to $y_p(x)$ where the c has been replaced with an unknown function $v(x)$. This will now be your guess for the *particular* solution. You can find $v(x)$ by plugging in your guess for the solution back into $y'(x)+p(x)y(x) = g(x)$.

3. Solve for $v(x)$.

4. Now that you know $v(x)$, the general solution to $y'(x) + p(x)y(x) = g(x)$ is given by $y_p(x)$ with $v(x)$ plugged in.

Exact Differential Equations and Integrating Factors

Definition: Exact Equation

An ODE of the form

$$\underbrace{M(x,y)}_{f_x} + \underbrace{N(x,y)}_{f_y} y' = 0,$$

is exact if and only if

$$(N(x,y))_x = (M(x,y))_y.$$

Alternatively, it might be easier to remember to check the conditions $f_{xy} = f_{yx}$, where f_x and f_y are indicated above.

If the ODE is exact, then $f(x,y)$ can be found by solving

$$f_x = M(x,y), \quad \text{and} \quad f_y = N(x,y).$$

Once $f(x,y)$ is determined, the implicit solution to the ODE is given by

$$\boxed{f(x,y) = c}$$

Definition: Nonlinear Integrating Factors (Nonlinear First Order ODE)

If an ODE of the form

$$N(x,y)y' + M(x,y) = 0,$$

is not exact, the goal is to find a function $\mu(x,y)$ so that when we multiply the ODE by $\mu(x,y)$ the equation becomes exact. In other words, can we find a $\mu(x,y)$ so that

$$\mu(x,y)N(x,y)y' + \mu(x,y)M(x,y) = 0,$$

is exact. This means that the nonlinear integrating function $\mu(x,y)$ satisfies the partial differential equation given by

$$(\mu(x,y)N(x,y))_x = (\mu(x,y)M(x,y))_y.$$

This can be a very difficult process. See the below for tricks.

Typically, we assume that $\mu(x,y)$ is a function of x or y ONLY. In doing so, we hopefully can find a solution for the function μ by solving the equation that results from $(\mu N)_x = (\mu M)_y$.

For example, if $\mu(x,y)$ is a function of x only $(\mu(x,y) \to \mu(x)$, then the above condition simplifies to

$$(\mu N)_x = (\mu M)_y \quad \Longrightarrow \quad \mu_x N + \mu N_x = \mu M_y$$

We can write this as a first order ODE for μ as

$$\mu_x = \mu \frac{(M_y - N_x)}{N} \tag{1}$$

and if $\frac{(M_y - N_x)}{N}$ depends only on x, we can solve this ODE for the integrating factor μ via separation of variables.

What would Equation (1) be if we assumed that μ was a function of only y? *Hint: expand* $(\mu N)_x = (\mu M)_y$ *using the product rule for derivatives remembering that* μ *does not depend on* x *so that* $\mu_x = 0$.

Bernoulli-Type ODEs

Definition: Bernoulli Type Differential Equation

If a first order ODE can be written in the form

$$y' + p(x)y = g(x)y^n,$$

where $n \neq 0, 1$, then the ODE can be transformed into a linear ODE for a new variable $v(x)$ by making the substitution $v(x) = (y(x))^{n-1}$. The function $v(x)$ can be solved for using standard linear techniques (linear integrating function or VOP), and $y(x)$ can be found letting $y(x) = (v(x))^{1/n}$.

Analyzing without Solving

We discussed several ways to analyze first order differential equations without solving. These methods include

- Examining the direction field - Let $y' = f(x, y)$. To draw the direction field,
  1. Choose a point in the x-y plane (x_0, y_0), and evaluate y' at this point.
  2. At the point (x_0, y_0), draw a small arrow with the slope found in part (1).
  3. Repeat this process for a lot of points (x_0, y_0). The plot that you obtain will give you a good idea of how solutions behave. *NOTE: You can use the dfield applet found at* http://math.rice.edu/ dfield/dfpp.html *to automatically generate this plot.*
- If $y' = f(y)$ (there is no dependence on x), we can plot y vs. y'. If y' is positive, the solution $y(x)$ is increasing, and if y' is negative, the solution is decreasing. This also gives us information regarding the behavior of solutions without solving the differential equation.

Existence and Uniqueness

Theorem

*Consider the following first-order, **linear**, ordinary differential equation subject to the initial conditions given:*

$$y' + p(x)y = g(x), \qquad y(x_0) = y_0,$$

*where $p(x)$ and $g(x)$ are continuous on an open interval I (i.e. $a < x < b$) that also contains x_0 (i.e. $a < x_0 < b$). Then, there exists exactly one solution $y(x) = \phi(x)$ of the differential equation $y' + p(x)y = g(x)$ **that also satisfies** the initial conditions $y(x_0) = y_0$. The solution is guaranteed to exist and be unique for all x in the interval I.*

Second Order Differential Equations (Homogeneous)

Constant Coefficient Problems

Consider the Linear 2nd order Constant Coefficient Homogeneous ODE (whew... that's a mouthful).

$$ay'' + by' + cy = 0$$

We can solve this differential equation with the following steps:

1. Assume that the solution is of the form $y = e^{\lambda x}$. Substitute this solution form into the differential equation to determine the characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

2. Solve the characteristic equation $a\lambda^2 + b\lambda + c = 0$ to find the roots or λ values. This equation has two roots, λ_1 and λ_2.
3. Write down the fundamental solutions:
   (a) λ_1 and λ_2 are both real and unique ($\lambda_1 \neq \lambda_2$). In this case, the two fundamental solutions are

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}$$

   (b) $\lambda_1 = \lambda_2$ (the case of real, repeated roots). Then the two fundamental solutions are

$$y_1 = e^{\lambda_1 x}, \quad y_2 = xe^{\lambda_1 x}$$

   (c) λ_1 and λ_2 are complex conjugates ($\lambda_1 = \alpha + i\beta$, and $\lambda_2 = \alpha - i\beta$). In this case, the two fundamental solutions are

$$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x),$$

4. Once you know the two fundamental solutions, you can write down the general solution

$$y = c_1 y_1 + c_2 y_2.$$

Existence & Uniqueness

Theorem

Consider the initial value problem

$$y'' + p(x)y' + q(x)y = g(x), \qquad y(x_0) = y_0, \quad y'(x_0) = y_0',$$

*where $p(x)$, $q(x)$ and $g(x)$ are continuous on an open interval I (*see note below) that also contains x_0. Then, there exists exactly one solution $y = \phi(x)$ of this problem and the solution exists throughout the interval I.*

*Note: I is just a range of x values where the functions p, q, and g are 11well behaved".

It's important to note the three things that this theorem says:

1. The initial value problem has *a* solution; in other words, a solution *exists.*
2. The initial value problem has *only one* solution; in other words, the solution is *unique*

3. The solution $\phi(x)$ is defined *throughout* the interval I where the coefficients are continuous and is at least twice differentiable there.

EXAMPLE

Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)\frac{d^2y}{dt^2} + t\frac{dy}{dt} - (t+3)y = 0, \qquad y(1) = 2, \qquad \frac{dy}{dt}\Big|_{t=1} = 1$$

SOLUTION

In this problem, if we write it in the form where the coefficient of the second derivative term is one, we find that $p(t) = 1/(t-3)$, $q(t) = -(t+3)/(t^2-3t)$, and $g(t) = 0$. The only points of discontinuity of the coefficients are at $t = 0$ and $t = 3$. Therefore, the longest open interval containing the initial point $t = 1$ in which all of the coefficients are continuous is $I = 0 < t < 3$. Thus, this is the longest interval in which our theorem guarantees that a solution exists.

THE WRONSKIAN, ABEL'S THEOREM, AND REDUCTION OF ORDER

DEFINITION: THE WRONSKIAN

The Wronskian of any two function $f(x)$ and $g(x)$ is given by

$$W(f,g)(x) = f(x)g'(x) - f'(x)g(x).$$

It can also be calculated as the determinant of the following matrix:

$$W(f,g)(x) = \det\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} = f(x)g'(x) - f'(x)g(x).$$

Theorem

*Linear Independence: Two functions $f(x)$ and $g(x)$ are **linearly dependent** if and only if their Wronskian*

$$W(f,g)(x) = f(x)g'(x) - f'(x)g(x) = 0,$$

for all x values.

Theorem

Abel's Theorem: Let y_1 and y_2 be any two fundamental solutions of

$$y'' + p(x)y' + q(x)y = 0,$$

then

$$W(y_1, y_2) = c \,\exp\left[-\int p(x)dx\right],$$

where c is an arbitrary, non-zero, constant.

Example

If you are given one fundamental solution (y_1) to the ODE

$$y'' + p(x)y' + q(x)y = 0,$$

how can you determine a second fundamental solution?

Solution

You can determine the second fundamental solution using either one of the following methods:

1. Use **Abel's Theorem** which states $W(y_1, y_2) = ce^{-\int p(x)\,dx}$ in conjunction with the definition of the Wronskian $W(y_1, y_2) = y_1y_2' - y_1'y_2$ to solve for y_2.
2. Assume that $y_2(x) = u(x)y_1(x)$. In other words, let your second fundamental solution be a function times your first. Plug in this guess to the ODE and solve for $u(x)$.

In class, we always used the first method, however, either method will work.

Second Order Differential Equations (Non-Homogeneous)

A linear second-order nonhomogeneous ODE is of the form

$$y'' + p(x)y' + q(x)y = g(x)$$

We discussed how the general-solution $y(x)$ consisted of two parts:

1. The homogeneous solution y_h. This solution solves the problem $y'' + p(x)y' + q(x)y = 0$.
2. The particular solution y_p. This solution solves the problem $y'' + p(x)y' + q(x)y = g(x)$.

Thus, $y(x) = y_h(x) + y_p(x)$. There are two different methods to solve the nonhomogeneous problem: (1) the method of undetermined coefficients, and $\boxed{\text{(2) variation of parameters}}$

Variation of parameters for 2nd order ODEs
Will not be covered on this exam.

Method of Undetermined Coefficients

If the differential equation is of the form

$$ay'' + by' + cy = g(x)$$

we can use MoUC. This method is generally pretty easy, however, there are down sides:

1. The coefficients of y'', y', and y must all be constants.
2. The function $g(x)$ must be of a specific form (see table below).

If the above conditions are true, then we can *guess* the form of the particular solution using the following rules:

Rule Name	If $g(x)$ is of the form...	Guess $y_p(x)$ of the form...
Constant	c	A
Exponential	$\alpha e^{\kappa x}$	$Ae^{\kappa x}$
Polynomial	$\alpha_N x^N + \ldots + \alpha_1 x + \alpha_0$	$A_N x^n + \ldots + A_1 x + A_0$
Exp-Poly	$e^{\kappa x}\left(\alpha_N x^N + \ldots + \alpha_1 x + \alpha_0\right)$	$e^{\kappa x}\left(A_N x^n + \ldots + A_1 x + A_0\right)$
Cosine-Sine	$\alpha\cos(\omega x) + \beta\sin(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$
Poly Cosine-Sine	$P_n(x)\cos(\omega x) + Q_m(x)\sin(\omega x)$	$S_N(x)\cos(\omega x) + T_N(x)\sin(\omega x)$
Poly-Exp Cosine-Sine	$e^{\kappa x}\left(P_n(x)\cos(\omega x) + Q_m(x)\sin(\omega x)\right)$	$e^{\kappa x}\left(S_N(x)\cos(\omega x) + T_N(x)\sin(\omega x)\right)$

So, in general, we use our seven rules to give an initial guess for the form of y_P. Next, if any term in this guess appears in the homogeneous solution, we multiply the entire guess by x. Now, we check if any of the new terms still appear in the homogeneous solution. If so, we multiply by x again, and so on.

THEREFORE, it is imperative that you know the fundamental solutions to the homogeneous problem before you make your guess for the particular solution.