

## Exam #2 Solutions

### Problem #1

a) If  $Y(s) = \frac{1}{s^4 + s^2}$ , then we can find  $y(t) = L^{-1}\{Y(s)\}$  via partial fractions or via the convolution.

Using partial fractions:

$$Y(s) = \frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

repeated

We don't even need to determine A, B, C, and D in order to determine that the answer is FALSE.

Given that  $Y(s)$  takes the form

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs}{s^2+1} + \frac{D}{s^2+1}$$

we can easily see that the inverse L.T. will take the form

$$y(t) = A + Bt + C \cdot \cos(t) + D \cdot \sin(t)$$

where A, B, C, and D are all constants. Thus,  $y(t)$  cannot take the form  $y(t) = t \cdot \sin(t)$ .

A quick calculation of the partial fraction decomposition shows

$$A=0, \quad B=1, \quad C=0, \quad D=-1$$

$$\therefore y(t) = t - \sin(t)$$

Final answer:

False.  $y(t) = t - \sin(t)$

b) If  $x_0=0$  is a R.S.P., then the following statements must be true for this problem.

$$\textcircled{1} P(1)=0 \quad \textcircled{2} \lim_{x \rightarrow 1} (x-1) \frac{Q(x)}{P(x)} \text{ is finite} \quad \textcircled{3} \lim_{x \rightarrow 1} (x-1)^2 \cdot \frac{R(x)}{P(x)} \text{ is finite.}$$

For this problem,  $P(x)=(x-1)^2$ ,  $Q(x)=x+1$ , and  $R(x)=1$

Checking the conditions, we have:

$$\textcircled{1} P(1)=(1-1)^2=0 \quad \checkmark$$

$$\begin{aligned} \textcircled{2} \lim_{x \rightarrow 1} (x-1) \cdot \frac{x+1}{(x-1)^2} &= \lim_{x \rightarrow 1} (x-1) \cdot \frac{x+1}{\cancel{(x-1)}} \cdot \frac{\cancel{(x-1)}}{(x-1)} \\ &= \lim_{x \rightarrow 1} \frac{x+1}{x-1} \rightarrow \frac{2}{0} \quad \text{this implies that} \\ &\quad \text{the limit D.N.E.} \quad \times \end{aligned}$$

Since  $x_0=1$  doesn't satisfy the conditions, the point  $x_0=1$  is not a regular singular point. However, it is a singular point..

Final answer:

FALSE. The point  $x_0=1$  is only a singular point.

Problem #2

a)

$$\begin{aligned}
 L\{u_1(t)(t^2+1)\} &= \lim_{A \rightarrow \infty} \int_0^A e^{-st} u_1(t)(t^2+1) dt \\
 &= \lim_{A \rightarrow \infty} \left[ \int_0^1 e^{-st} \cdot 0 \cdot (t^2+1) dt + \int_1^A e^{-st} \cdot 1 \cdot (t^2+1) dt \right] \\
 &= \lim_{A \rightarrow \infty} \int_1^A e^{-st} (t^2+1) dt \\
 &= \lim_{A \rightarrow \infty} \left[ -\frac{(t^2+1)}{s} e^{-st} - \frac{2t}{s^2} e^{-st} - \frac{2}{s^3} e^{-st} \right]_1^A \\
 &= \lim_{A \rightarrow \infty} \left( -\frac{(A^2+1)}{s} e^{-sA} - \frac{2A}{s^2} e^{-sA} - \frac{2}{s^3} e^{-sA} \right) \underset{s>0}{\text{provided}} \\
 &\quad - \left( -\frac{2}{s} e^{-s} - \frac{2}{s^2} e^{-s} - \frac{2}{s^3} e^{-s} \right) \\
 &= \left( \frac{2}{s} + \frac{2}{s^2} + \frac{2}{s^3} \right) e^{-s}
 \end{aligned}$$

since  $u_1(t) = \begin{cases} 0 & t < 1 \\ 1 & t \geq 1 \end{cases}$   
split the integral !!

Side calculation:  $\int e^{-st} (t^2+1) dt$   
using tabular integration by parts.

$$\begin{array}{c|c}
 U^{(n)} & V^{(-n)} \\
 \hline
 t^2+1 & e^{-st} \\
 2t & -\frac{1}{s} e^{-st} \\
 2 & \frac{1}{s^2} e^{-st} \\
 0 & -\frac{1}{s^3} e^{-st}
 \end{array}$$

Final Answer:

$$L\{u_1(t)(t^2+1)\} = e^{-s} \cdot \left( \frac{2}{s} + \frac{2}{s^2} + \frac{2}{s^3} \right)$$

- b) If the solution can be expressed as a Taylor series about  $x_0=1$  then  $x_0=1$  is an ordinary point and  $P(1) \neq 0$ .

Furthermore, since the radius of convergence is 2 (given by  $|x-x_0|<2$ ) the closest singularity is 2 units away (@  $x=3$  or  $x=-1$  for example)

Lets choose the closest singularity to be at  $x=3$ . Thus  $P(3)=0$

A sample differential equation would be

$$(x-3)y'' + xy' + y = 0$$

### Problem #3

a)  $\mathcal{L}\{y'' + 7y' + 12y\} = \mathcal{L}\{\sin(\frac{\pi t}{2}) \delta(t-1)\}$  where  $y(0)=y'(0)=0$   
 $s^2Y(s) + 7Y(s) + 12Y(s) = \int_0^\infty e^{-st} \sin(\frac{\pi t}{2}) \delta(t-1) dt$  use the definition  
 $(s^2 + 7s + 12)Y(s) = e^{-s} \sin(\frac{\pi}{2})$

$$Y(s) = \frac{e^{-s}}{s^2 + 7s + 12}$$

call this H(s)

$$Y(s) = e^{-s} \cdot \left( \frac{1}{(s+3)(s+4)} \right)$$

By rule #13,  $y(t) = u_1(t) \cdot h(t-1)$  where  $h(t) = \mathcal{L}^{-1}\{H(s)\}$ , so we just need to figure out  $h(t)$ .

Side calculations :

$$H(s) = \frac{1}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$1 = A(s+4) + B(s+3) \Rightarrow B=-1, A=1$$

$$H(s) = \frac{1}{s+3} - \frac{1}{s+4}$$

$$\therefore h(t) = e^{-3t} - e^{-4t}$$

Now, using our side calculation for  $h(t)$ , we have our final answer:

$$y(t) = u_1(t) \cdot \left( e^{-3(t-1)} - e^{-4(t-1)} \right)$$

b) First, note that  $g(t) = 3 \cdot (u_3(t) - u_6(t))$

$$s^2 Y(s) - s + 1 + 9Y(s) = L\{3 \cdot (u_3(t) - u_6(t))\}$$

$$(s^2 + 9)Y(s) = s - 1 + \frac{3}{s} (e^{-3s} - e^{-6s})$$

Solving for  $Y(s)$ , we get

$$Y(s) = \frac{s-1}{s^2+9} + \frac{3}{s(s^2+9)} (e^{-3s} - e^{-6s})$$

or

$$Y(s) = \frac{s}{s^2+9} - \frac{1}{s^2+9} + (e^{-3s} - e^{-6s}) \cdot \frac{3}{s(s^2+9)}$$

call this  
H(s) for simplicity

$$\therefore y(t) = \cos(3t) - \frac{1}{3} \sin(3t) + u_3(t)h(t-3) - u_6(t)h(t-6)$$

where  $h(t) = L^{-1}\{H(s)\}$ .

Side calculation:

$$H(s) = \underbrace{\frac{1}{s}}_{F(s)} \cdot \underbrace{\frac{3}{s^2+9}}_{G(s)} \rightarrow \text{perfect candidate for convolution.}$$

$\downarrow \quad \downarrow$

$f(t) = 1 \quad g(t) = \sin(3t)$

$$\begin{aligned} h(t) &= 1 * \sin(3t) \\ &= \int_0^t 1 \cdot \sin(3\tau) d\tau \\ &= -\cos(3\tau) \Big|_0^t \\ &= -\cos(3t) - (-\cos(0)) \\ &= \underline{1 - \cos(3t)} \end{aligned}$$

Using our side calculation, we have

$$y(t) = \cos(3t) - \frac{1}{3} \sin(3t) + u_3(t) \cdot (1 - \cos(3(t-3))) - u_6(t) \cdot (1 - \cos(3(t-6)))$$

### Problem #4

a) Ordinary points:  $P(x) \neq 0$ . Since  $P(x) = x$ , the ordinary points are all  $x$  such that  $x \neq 0$ .

Singular points:  $P(x) = 0$ . The only singular point is  $x=0$ . We should check to see if this is a regular singular pt.

$$\textcircled{1} \quad P(0) = 0 \quad \checkmark$$

$$\textcircled{2} \quad \lim_{x \rightarrow 0} x \cdot \frac{Q(x)}{P(x)} = \lim_{x \rightarrow 0} x \cdot \frac{1}{x} = 1 \quad \checkmark$$

$$\textcircled{3} \quad \lim_{x \rightarrow 0} x^2 \cdot \frac{R(x)}{P(x)} = \lim_{x \rightarrow 0} x^2 \cdot \frac{1}{x} = 0 \quad \checkmark$$

$\therefore x=0$  is a regular singular point.

The pt.  $x=0$  is the only singular point and is a regular singular point. All other  $x$  values (specifically  $x \neq 0$ ) are ordinary points.

b) Since  $x=1$  is an ordinary point, we can express the series solution to the ODE in the following form:

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y'(x) = \sum a_n (n)(x-1)^{n-1}$$

$$y''(x) = \sum a_n (n)(n-1)(x-1)^{n-2}$$

} plug these into the ODE

But first, we need to put the ODE in a form that's easy to work with:

$$(x-1+1)y'' + y' + y = 0$$

$$(x-1)y'' + y'' + y' + y = 0$$

$$(x-1) \sum_{n=2}^{\infty} a_n (n)(n-1)(x-1)^{n-2} + \sum_{n=2}^{\infty} a_n (n)(n-1)(x-1)^{n-2} \\ + \sum_{n=1}^{\infty} a_n n(x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Combining and reindexing...

$$\sum_{n=1}^{\infty} a_{n+1} (n+1)n(x-1)^n + \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1)(x-1)^n \\ + \sum_{n=0}^{\infty} a_{n+1} (n+1)(x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Rewriting the terms

$$a_2(2)(1)(x-1)^0 + a_1(1)(x-1)^0 + a_0(x-1)^0 + \\ + \sum_{n=1}^{\infty} [(n+1)n a_{n+1} + (n+2)(n+1) a_{n+2} + a_{n+1}(n+1) + a_n] (x-1)^n = 0$$

equating like powers of  $x-1$ :

$$(x-1)^0: \quad 2a_2 + a_1 + a_0 = 0 \quad \rightarrow \quad a_2 = -\frac{1}{2}(a_0 + a_1)$$

$$(x-1)^1: \quad (n+2)(n+1)a_{n+2} + (n+1)(n+1)a_{n+1} + a_n = 0$$

$$\hookrightarrow a_{n+2} = -\frac{a_n + (n+1)^2 a_{n+1}}{(n+1)(n+2)}, \quad n \geq 1$$

recursion relationship.

Writing out the first few terms:

$$n=1: \quad a_3 = -\frac{a_1 + 2^2 a_2}{2 \cdot 3} = -\frac{a_1}{6} - \frac{4}{6} a_2 \\ = -\frac{a_1}{6} - \frac{2}{3} \left( -\frac{1}{2}(a_0 + a_1) \right)$$

$$a_3 = -\frac{a_1}{6} + \frac{1}{3}(a_0 + a_1) = \frac{1}{3}a_0 + \frac{1}{6}a_1$$

$$y(x) = a_0 + a_1(x-1) - \frac{1}{2}(a_0 + a_1)(x-1)^2 + \left(\frac{1}{3}a_0 + \frac{1}{6}a_1\right)(x-1)^3 + \text{h.o.t.}$$

regrouping...

$$y(x) = a_0 \left( 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \text{h.o.t.} \right) + a_1 \left( (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \text{h.o.t.} \right)$$

So the fundamental solutions are given by

$$y_1(x) = 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \text{h.o.t.}$$

$$y_2(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \text{h.o.t.}$$