# TRUE / FALSE

For the **True / False** questions **you must justify your answers.**

1. (4 points) **True / False.** It is possible to find an integrating function $\mu(y)$ ($\mu$ depends only on $y$) such that the following differential equation

$$x^2 + 2x + y^2 + 2yy' = 0 \quad (*)$$

becomes exact when multiplied by $\mu$.

If we multiply $(*)$ by $\mu(y)$, we get

$$\underbrace{\mu(y)\left[x^2 + 2x + y^2\right]}_{M} + \underbrace{2y\,\mu(y)}_{N}\,y' = 0$$

To see if the new ODE is exact, we check: does $M_y = N_x$?

$$M_y = \mu'(y)\left[x^2 + 2x + y^2\right] + \mu(y)\left[2y\right]$$

$$N_x = 0$$

Can we find $\mu(y)$? No. If we solve

$$\mu'(y)\left[x^2 + 2x + y^2\right] + 2y\mu(y) = 0$$

then $\mu(y)$ would also depend on $x$.

$\boxed{\text{FALSE}}$

2. (4 points) **True / False.** There exists a unique solution to the following initial value problem

$$xy' = y, \qquad y(0) = 0.$$

The ODE is separable, so let's try!

$$x \cdot \frac{dy}{dx} = y \longrightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \longrightarrow \ln|y| = \ln|x| + \tilde{c}$$

So, we have $y = cx$. Using the initial conditions, we have

$$0 = c \cdot 0 \longrightarrow c = \text{anything!}$$

Thus, $y = cx$ represents an infinite class of solutions to the IVP. And $\therefore$ there is not a unique solution.

$\boxed{\text{FALSE}}$

Points earned: ______________ out of a possible 8 points

# MATCHING/MULTIPLE CHOICE

3. Consider the plots of solutions to initial value problems for various second-order differential equations of the form

$$ay'' + by' + cy = g(x),$$

(plots given on the next page).

If possible, match the following differential equations with a plot that best matches a solution to a corresponding initial value problem. If it is not possible to match the differential equation with a corresponding figure, use one of the blank grids on the next page to sketch a suitable graph corresponding to a non-trivial (non-zero) solution to a corresponding initial value problem.

(a) (4 points) $y'' - 2y' + 2y = 0$

$\lambda^2 - 2\lambda + 2 = 0$

$\lambda - 2\lambda + 1 + 1 = 0$

$(\lambda - 1)^2 = -1$

$\underline{\underline{\lambda = 1 \pm i}}$

$y_1 = e^x \cos(x)$

$y_2 = e^x \sin(x)$

$y(x) = e^x \sqrt{c_1^2 + c_2^2} \cos(x - \theta)$

**A solution corresponds to Figure** C

(b) (4 points) $3y'' + 17y' + 10y = 0$

$3\lambda^2 + 17\lambda + 10 = 0$

$(3\lambda + 2)(\lambda + 5) = 0$

$\lambda = -\frac{2}{3}$ or $\lambda = -5$

$y = c_1 e^{-2/3 x} + c_2 e^{-5x}$

as $x \to \infty$ $y \to 0$ !

**A solution corresponds to Figure** E

(c) (4 points) $y'' + y = \sin(2t)$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$y_h = c_1 \cos(t) + c_2 \sin(t)$

$y_p = A\cos(2t) + B\cos(2t)$

**A solution corresponds to Figure** A

Points earned: ______ out of a possible 12 points

Figure A

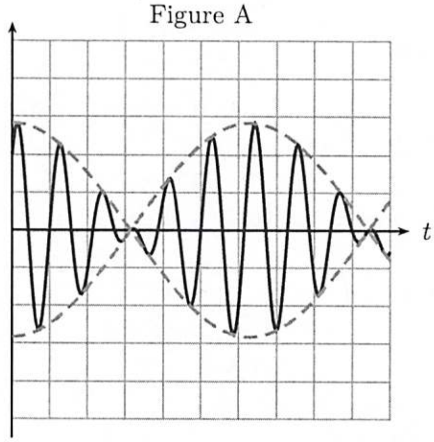



Figure B

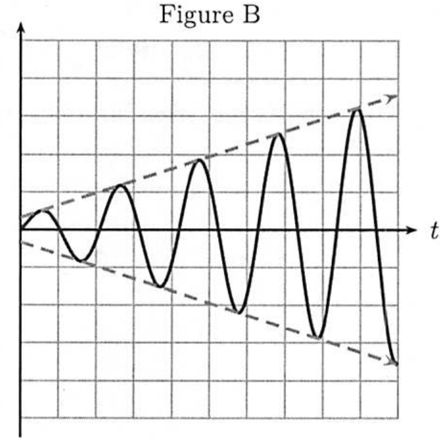



Figure C

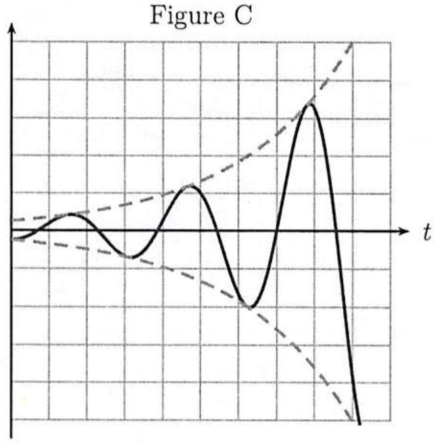



Figure D

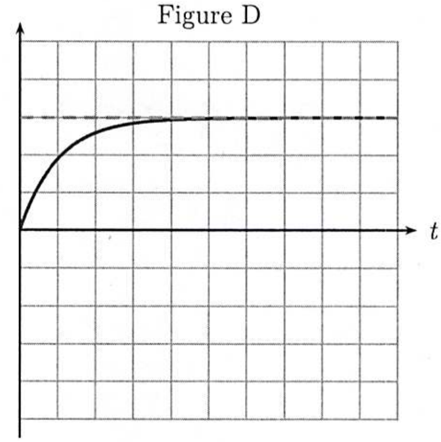



Figure E

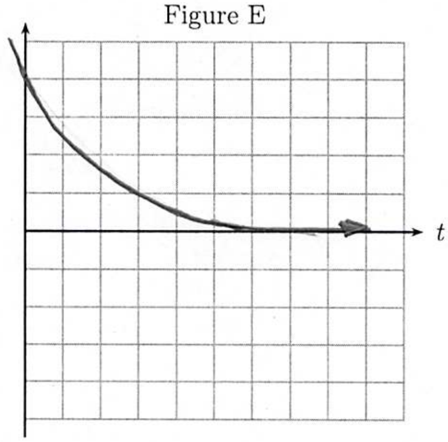
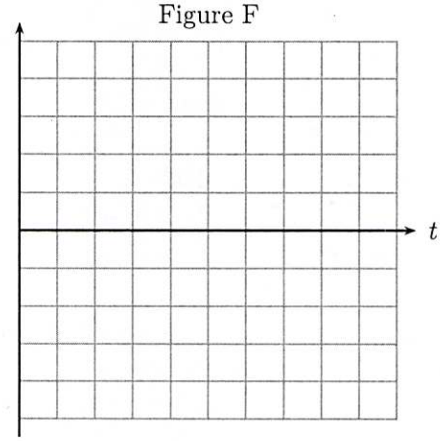



Figure F




Points earned: ____________ out of a possible 0 points

## FIND THE SOLUTION

Find the general solution to each of the following differential equations.

**You may leave your solution in implicit form if you like.**

4. (10 points) $4y'' - 8y' - 4y = 0$

Assume $y = e^{\lambda x} \longrightarrow$ char. eqn. $\longrightarrow 4\lambda^2 - 8\lambda - 4 = 0$

$4(\lambda^2 - 2\lambda - 1) = 0$

$\lambda - 2\lambda + 1 - 2 = 0$

$(\lambda - 1)^2 = 2 \qquad \lambda = 1 \pm \sqrt{2}$

$y_1 = e^{(1+\sqrt{2})x} \longrightarrow \boxed{y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x}}$

$y_2 = e^{(1-\sqrt{2})x}$

5. (10 points) $r'(t) = \dfrac{t - e^{-t}}{r(t) + e^{r(t)}}$

Separable!

$$\int (r + e^{r})\,dr = \int t - e^{-t} \quad dt$$

$$\boxed{\frac{1}{2}r^{2} + e^{r} = \frac{1}{2}t^{2} + e^{-t} + c}$$

Points earned: ________ out of a possible 10 points

6. (15 points) $y'' + 2y' + y = \frac{e^{-t}}{t}$

In order to solve this ODE, you would need to use variation of parameters for a second order ODE. This topic will not be included on the Winter 2015 Exam #1.

2nd order, linear, const coeff. non-homog!

$\boxed{\text{Find } y_h}$

$y'' + 2y' + y = 0 \longrightarrow \lambda^2 + 2\lambda + 1 = 0$

$\lambda = -1$ repeated

$\underline{y_1 = e^{-t}}$ $\underline{y_2 = te^{-t}}$

$\boxed{\text{Find } y_p}$ Must use VOP

$$W(y_1, y_2) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{vmatrix} = e^{-2t}[1 - t + t] = e^{-2t} \checkmark$$

$\underline{W = e^{-2t}}$

$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$ where $u_1 = \int \frac{-y_2 g(x)}{W} dx$ $u_2 = \int \frac{y_1 g(x)}{W}$

$$u_1 = \int \frac{-te^{-t} \cdot \frac{1}{t}e^{-t}}{e^{-2t}} dt = \int -dt = -t$$

$$u_2 = \int \frac{e^{-t} \cdot \frac{1}{t}e^{-t}}{e^{-2t}} dt = \int \frac{1}{t} dt = \ln|t|$$

So...

$$\boxed{y(x) = c_1 e^{-t} + c_2 te^{-t} - te^{-t} + \ln|t| te^{-t}}$$

Points earned: ____________ out of a possible 15 points

Solve each of the following initial value problems.

**You may leave your solution in implicit form if you like.**

7. (15 points) Solve the initial value problem $xy' = y + x^2e^{-x}$, $y(1) = 4$.

$y' - \frac{1}{x}y = xe^{-x}$ $\qquad y(1) = 4.$

$\mu = e^{-\int \frac{1}{x}dx}$

$= e^{-\ln|x|} = \frac{1}{x}$

$\frac{d}{dx}\left[\frac{1}{x}\cdot y(x)\right] = e^{-x}$

$\frac{1}{x}y(x) = -e^{-x} + c$ $\qquad$ if $x=1, y=4$

$\frac{1}{1}\cdot 4 = -e^{-1} + c$ $\qquad \rightarrow \quad c = 4 + e^{-1}$

$\boxed{\frac{1}{x}y(x) = e^{-x} + e^{-1} + 4}$

Points earned: __________ out of a possible 15 points

8. (15 points) $\dfrac{dy}{dx} = \dfrac{3x + xy^2}{y + x^2y}, \qquad y(1) = 3$

$$\frac{dy}{dx} = \frac{x(3+y^2)}{y(1+x^2)}$$ separable!

$$\int \frac{y}{3+y^2}\,dy = \int \frac{x}{1+x^2}\,dx \rightarrow \frac{1}{2}\ln(3+y^2) = \frac{1}{2}\ln(1+x^2) + c$$

$$\cancel{\tfrac{1}{2}}\ln(12) = \frac{1}{2}\ln(2) + c$$

$$c = \frac{1}{2}\ln(6) \rightarrow c = \ln(\sqrt{6})$$

$$\boxed{\frac{1}{2}\ln(3+y^2) = \frac{1}{2}\ln(1+x^2) + \ln(\sqrt{6})}$$

9. (15 points) $q'' + 4q' + 13q = e^{-2t}, \quad q(0) = 1,\ q'(0) = 2$

$\lambda^2 + 4\lambda + 4 + 9 = 0$

$\lambda = -2 \pm 3i$

$q_1 = e^{-2t}\cos(3t)$

$q_2 = e^{-2t}\sin(3t)$

$q_p = Ae^{-2t}$ $\quad q_p' = -2Ae^{-2t}$

$q_p'' = 4Ae^{-2t}$

$4A - 8A + 13A = \cancel{0}\ 1$

$9A = 1$ $\quad \boxed{A = \frac{1}{9}}$

$q = e^{-2t}\left(c_1\cos(3t) + c_2\sin(3t)\right) + \frac{1}{9}e^{-2t}$

$q' = -2e^{-2t}\left(c_1\cos(3t) + c_2\sin(3t) + \frac{1}{9}\right) + e^{-2t}\left[-c_1\sin(3t)\cdot 3 + 3c_2\cos(3t)\right]$

$q(0) = 1 \rightarrow c_1 + \frac{1}{9} = 1$ $\quad \underline{c_1 = \frac{8}{9}}$

$q'(0) = 2 \rightarrow -2\left(c_1 + \frac{1}{9}\right) + \left[3c_2\right] = 2$

$-2 + 3c_2 = 2$

$3c_2 = 4 \rightarrow c_2 = \frac{3}{4}$

$$\boxed{q(t) = e^{-2t}\cdot\left[\frac{8}{9}\cos(3t) + \frac{3}{4}\sin(3t) + \frac{1}{9}\right]}$$