Conservation Laws for ~~the~~ Water Waves

A Different Perspective

Katie Oliveras

Seattle University — Department of Mathematics

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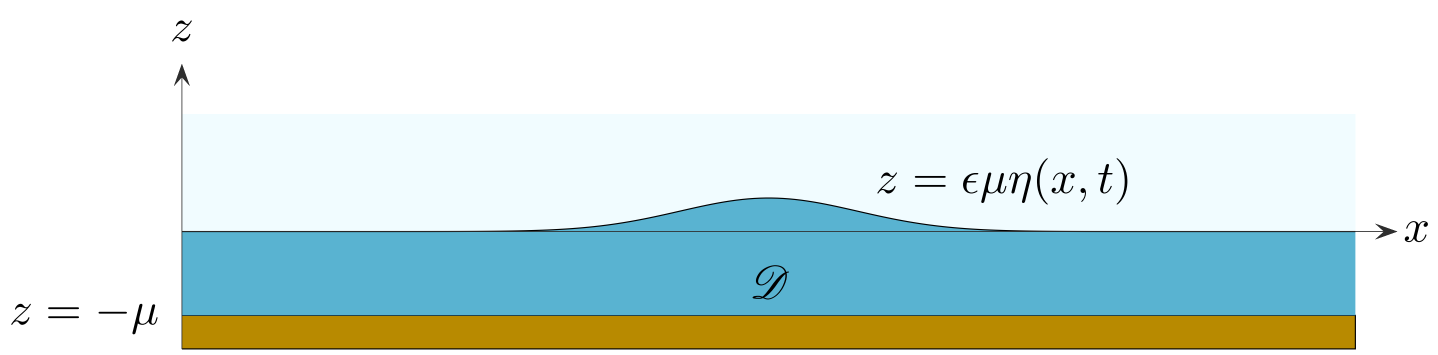
Equations of Motion

We begin by considering an inviscid, irrotational, fluid with a one-dimensional free-surface on the whole-line:

$$
\begin{aligned}
&\phi_{xx} + \phi_{zz} = 0, && (x,z) \in \mathscr{D}, \\
&\phi_z = 0, && z = -\mu, \\
&\mu\eta_t = \phi_z - \epsilon\mu\eta_x\phi_x, && z = \epsilon\mu\eta(x,t), \\
&\phi_t + \frac{\epsilon}{2}\left(\phi_x^2 + \phi_z^2\right) + \mu\eta = 0, && z = \epsilon\mu\eta(x,t),
\end{aligned}
$$

where $\phi(x,z,t)$ represents the velocity potential of the fluid, $\eta(x,t)$ represents the surface elevation.



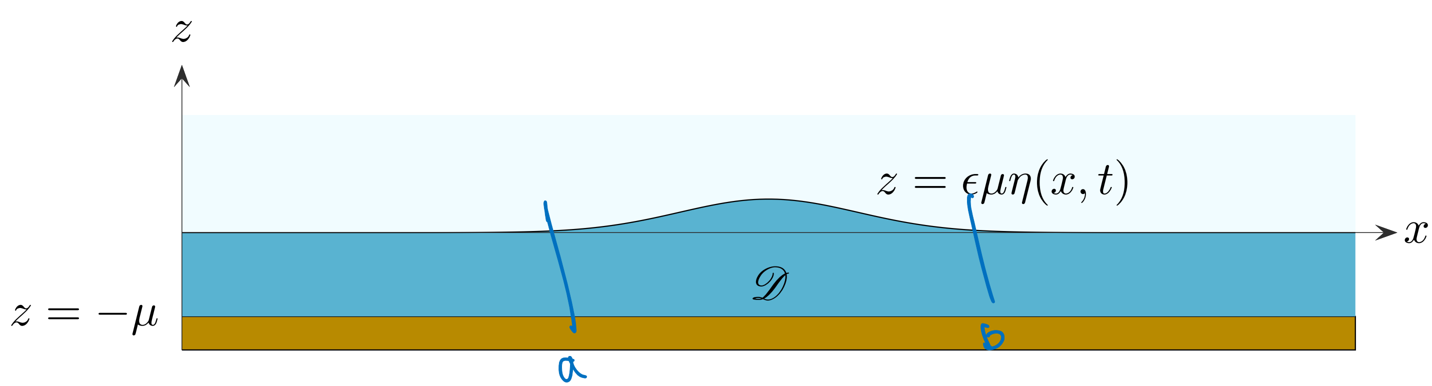
What *is* a conservation law?

Following the definition of **[Benjamin & Olver]**, T is a **conserved density** if there exist function P, Q and W such that

$$\frac{d}{dt}\int_a^b T\Big|_{z=\epsilon\mu\eta} dx = \int_a^b \left((P + W_x) + \epsilon\mu\eta_x(W_z - Q)\right)\Big|_{z=\epsilon\mu\eta} dx$$

with

$$\overset{Q}{\cancel{U}}_z + \overset{P}{\cancel{V}}_x = 0 \text{ in } \mathscr{D}$$



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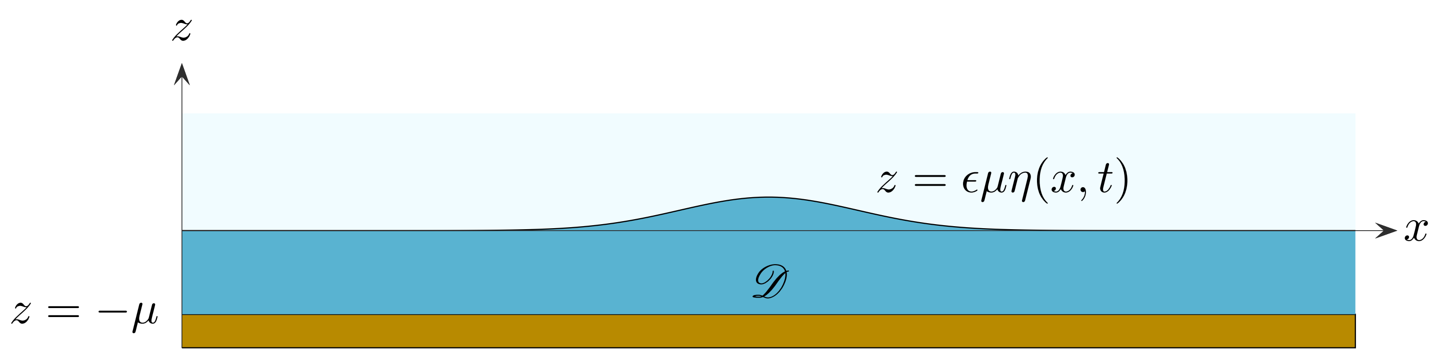
with

typo

$$U_z + V_x = 0 \text{ in } \mathscr{D}$$

Note that this implies that there may be contributions from the bottom and sides.



Prior Results of Benjamin & Olver

The eight conservation laws are found **[Benjamin & Olver, Olver]**

$$T_1 = -\eta_x \Phi \qquad T_2 = \frac{1}{2}\Phi\Phi_{(n)} + \frac{1}{2}g\eta^2$$

$$T_3 = \eta \qquad T_4 = \Phi + gt\eta$$

$$T_5 = x\eta + t\eta_x\Phi \qquad T_6 = \frac{1}{2}\eta^2 - t\Phi - \frac{1}{2}gt^2\eta$$

$$T_7 = (\eta - x\eta_x)\Phi - t(4T_2 - \frac{7}{2}g\eta^2) - \frac{7t^2}{2}g\phi - \frac{7t^3}{6}g^2\eta \qquad T_8 = (x + \eta\eta_x)\Phi + tgx\eta + \frac{1}{2}gt^2\eta_x\Phi$$

mention challenge with computing T_7.

- Olver proved that these are the only conservation laws that arise from the infinitesimal Lie symmetries of the problem.
- The proof is only for a one-dimensional surface. There is no such proof for the 3d problem.
- Olver began with bulk quantities and then restricted to the free-surface. What would happen if we started at the free surface?

Equations of Motion - Velocity Potential Formulation

What are the various ways to transform these equations into surface variables?

$$
\begin{aligned}
&\phi_{xx} + \phi_{zz} = 0, && (x,z) \in \mathscr{D},\\
&\phi_z = 0, && z = -\mu,\\
&\mu\eta_t = \phi_z - \epsilon\mu\eta_x\phi_x, && z = \epsilon\mu\eta(x,t),\\
&\phi_t + \frac{\epsilon}{2}\left(\phi_x^2 + \phi_z^2\right) + \mu\eta = 0, && z = \epsilon\mu\eta(x,t),
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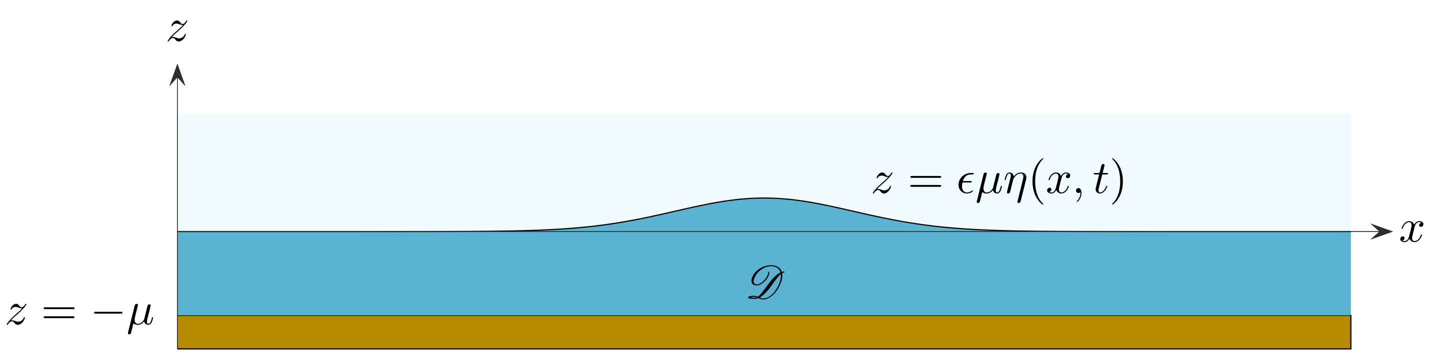


Equations of Motion - ZCS Formulation

[Zakharov, and Craig & Sulem (ZCS)]:

$$\boxed{q(x,t) = \phi(x, \epsilon\mu\eta(x,t), t)}$$

Dynamic Boundary Condition

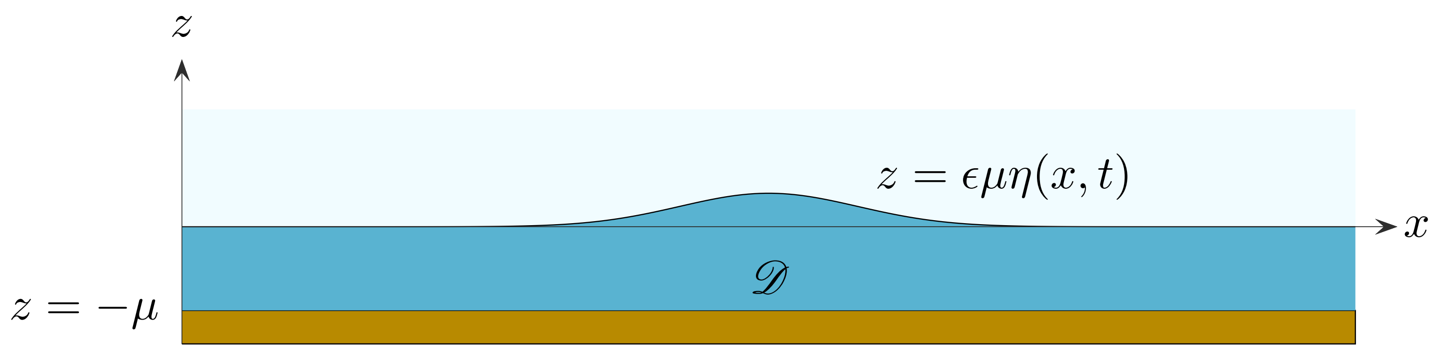
$$q_t + \frac{\epsilon}{2} q_x^2 + \mu\eta - \frac{1}{2}\frac{\left(\mathcal{G}(\eta)q + \eta_x q_x\right)^2}{1+\eta_x^2} = 0,$$

TYPO
non dimensional forms are incorrect. fix this!

Kinematic Boundary Condition

$$\eta_t = \mathcal{G}(\eta)q,$$

where $\mathcal{G}(\eta)q$ is the Dirichlet-to-Neumann operator $\mathcal{G}(\eta)q = \nabla\phi \cdot \vec{n}$.



Equations of Motion - AFM Formulation

[Ablowitz, Fokas, & Musslimani (AFM)]:

$$\boxed{q(x,t) = \phi(x, \epsilon\mu\eta(x,t), t)}$$

Local Equation

$$q_t + \frac{\epsilon}{2}q_x^2 + \mu\eta - \frac{1}{2}\frac{(\eta_t + \eta_x q_x)^2}{1 + \eta_x^2} = 0,$$

$\epsilon^2\mu^2$

TYPO

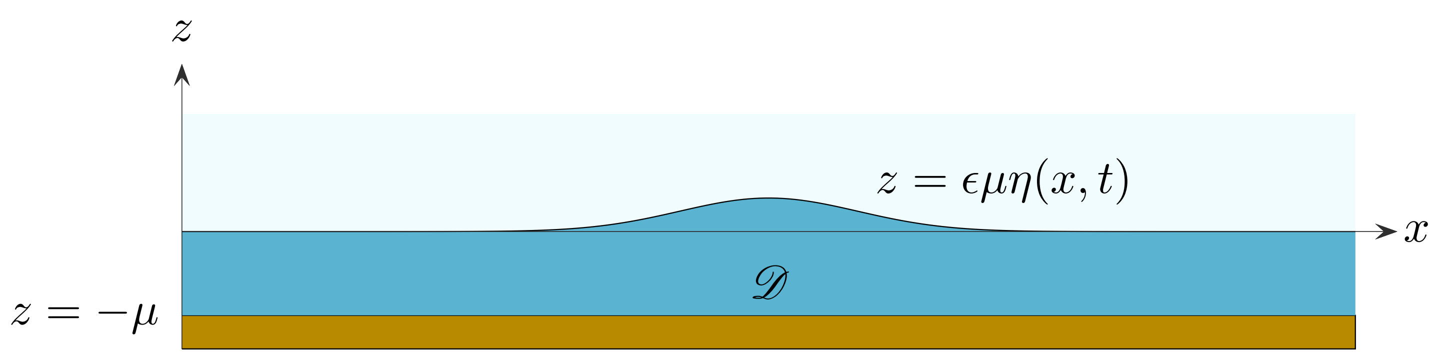
non-dimensional form is incorrect. fix this

Nonlocal Equation

G(η)q

ε missing

$$\int_{-\infty}^{\infty} \left(e^{-ikx}\left(\mu\eta_t \cosh(\mu k(\epsilon\eta + 1)) + iq_x \sinh(\mu k(\eta + 1))\right)\right) dx = 0$$



Equations of Motion - AFM Formulation

[Ablowitz, Fokas, & Musslimani (AFM)]:

$$\boxed{q(x,t) = \phi(x, \epsilon\mu\eta(x,t), t)}$$

Local Equation (Dynamic BC)

Typo Nondimensional form is incorrect. Fix this!

$$q_t + \frac{\epsilon}{2}q_x^2 + \mu\eta - \frac{1}{2}\frac{(\eta_t + \eta_x q_x)^2}{1+\eta_x^2} = 0,$$

Nonlocal Equation (Kinematic BC)

$$\int_{-\infty}^{\infty} \left(e^{-ikx}\left(\mu\eta_t \cosh(\mu k(\epsilon\eta + 1)) + iq_x \sinh(\mu k(\eta + 1))\right)\right)\, dx = 0$$

missing an ϵ

Note when $k = 0$, the **nonlocal equation** yields a conservation law:

$$\int_{-\infty}^{\infty} (\mu\eta_t)\, dx = 0 \qquad \text{(T3)}$$

They also find (T5) via an asymptotic expansion of the **nonlocal equation** as $k \to 0$ in combination with the local equation.

Conservation Laws Hiding in Plain Sight

Nonlocal Equation

$$\int_{-\infty}^{\infty} \left(e^{-ikx}\left(\mu\eta_t \cosh(\mu k(\epsilon\eta+1)) + iq_x \sinh(\mu k(\epsilon\eta+1))\right)\right)\, dx = 0$$

The conservation laws (T3) and (T5) are *hiding in plain sight* in the nonlocal equation.

Motivating Question

Are there other conservation laws hiding within the nonlocal equation?

In what follows, we will

1. develop a nonlocal-nonlocal formulation
2. search for conservation laws
3. generalize to other scenarios

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Nonlocal/Nonlocal Formulation: The First Integral Relation

For **any harmonic function** $\varphi(x, z)$, we have

$$\oint_{\partial \mathscr{D}} \left((\varphi_z \nabla \phi - \phi \nabla \varphi_z) \cdot \mathbf{n} \right) ds = 0,$$

→ Divergence thm yields

$$= \iint_D \nabla\varphi_z \cdot \nabla\phi - \nabla\phi \cdot \nabla\varphi_z \, dz\, dx$$

$$= 0$$

where we assume that ϕ has sufficient decay properties as $|x| \to \infty$.

Nonlocal/Nonlocal Formulation: The First Integral Relation

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$$\int_{\mathscr{S}} \varphi_z \epsilon\mu\eta_t \, dx = \oint_{\partial\mathscr{D}} \left(\epsilon\phi\nabla\varphi_z \cdot \mathbf{n} \right) \, ds$$

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Notation

We will use the following notation

$$\int_{\mathscr{S}} F \, dx = \int_{-\infty}^{\infty} \left(F(x, \epsilon\mu\eta, t) \right) \, dx, \qquad \int_{\mathscr{B}} F \, dx = \int_{-\infty}^{\infty} \left(F(x, -\cancel{\mu}, t) \right) \, dx$$

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$$\downarrow$$

$$\boxed{\int_{\mathscr{S}} \frac{d}{dt}\varphi \, dx = \oint_{\partial\mathscr{D}} \left(\epsilon\phi\nabla\varphi_z \cdot \mathbf{n} \right) \, ds}$$

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$$\downarrow$$

$$\boxed{\int_{\mathscr{S}} \frac{d}{dt}\varphi \, dx = \oint_{\partial\mathscr{D}} \left(\epsilon\phi\nabla\varphi_z \cdot \mathbf{n}\right) \, ds}$$

Recovering AFM

If we let $\varphi = e^{-ikx}\sinh(k(z+\mu))$, then we recover the **AFM nonlocal equation**.

Nonlocal/Nonlocal Formulation: The Second Integral Relation

Since ϕ is harmonic in $\mathscr{D}$, then so is ϕ_t.

Nonlocal/Nonlocal Formulation: The Second Integral Relation

Since ϕ is harmonic in $\mathscr{D}$, then so is ϕ_t. Like before, for **any harmonic function $\varphi(x,z)$**, we have

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$$\oint_{\partial\mathscr{D}} ((\varphi_z \nabla\phi_t - \phi_t \nabla\varphi_z)\cdot\mathbf{n})\, ds = 0,$$

$$\downarrow$$

$$\frac{d}{dt}\int_{\mathscr{S}} \left(q\frac{\partial\varphi_z}{\partial\mathbf{n}}\right) dx = \int_{\mathscr{S}} (q_t\varphi_{zz} + \epsilon\mu(\eta_t q_x - \eta_x q_t)\varphi_{xz})\, dx$$

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$$\downarrow$$

$$\frac{d}{dt}\int_{\mathscr{S}} \left(q \frac{\partial \varphi_z}{\partial \mathbf{n}} \right) \, dx \quad = \quad \int_{\mathscr{S}} \left(-q_t\varphi_{zz} + \epsilon\mu(\eta_t q_x - \eta_x q_t)\varphi_{xz} + 2q_t\varphi_{zz} \right) \, dx$$

Nonlocal/Nonlocal Formulation: The Second Integral Relation

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$$\downarrow$$

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Nonlocal/Nonlocal Formulation: The Second Integral Relation

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$$\oint_{\partial\mathscr{D}} ((\varphi_z \nabla\phi_t - \phi_t \nabla\varphi_z)\cdot\mathbf{n})\, ds = 0,$$

$$\downarrow$$

$$\frac{d}{dt}\int_{\mathscr{S}} \left(q \frac{\partial \varphi_z}{\partial \mathbf{n}} \right) dx = \int_{\mathscr{S}} (-q_t\varphi_{zz} + \epsilon\mu(\eta_t q_x - \eta_x q_t)\varphi_{xz} + 2q_t\varphi_{zz})\, dx$$

Why Split?

For any harmonic function φ,

$$\int_{\mathscr{S}} \underline{-\epsilon q_t\varphi_{zz} + \epsilon^2\mu(\eta_t q_x - \eta_x q_t)}\varphi_{xz}\, dx = \oint_{\partial\mathscr{D}} \underline{\left(\phi_t + \frac{1}{2}|\nabla\phi|^2\right)} (\varphi_{zz}\, \underline{dx + \varphi_{xz}\, dz}) - \int_{\mathscr{B}} \epsilon\phi_t\varphi_{zz}\, dx$$

note that this is only using $\mu\eta_t = \nabla\Phi\cdot n$ (kinematic)

Nonlocal/Nonlocal Formulation: The Second Integral Relation

Since ϕ is harmonic in $\mathscr{D}$, then so is ϕ_t. Like before, for **any harmonic function** $\varphi(x,z)$, we have

$$\oint_{\partial\mathscr{D}} \left((\varphi_z \nabla\phi_t - \phi_t \nabla\varphi_z)\cdot\mathbf{n}\right)\, ds = 0,$$

$$\downarrow$$

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dynamic

Finally, using the ~~kinematic~~ **boundary condition** and some suggestive IBP, the resulting relationship becomes

$$\boxed{\int_{\mathscr{S}} \frac{d}{dt} q\left(\frac{\partial\varphi_z}{\partial\mathbf{n}} - \varphi_{zz}\right)\, dx = \int_{\mathscr{S}} \mu\eta\left(\varphi_{zz} + \epsilon\mu\eta_x\varphi_{xz}\right) - 2\epsilon\mu\eta_t\, q\, \varphi_{zzz}\, dx + \int_{\mathscr{B}} \frac{\epsilon}{2} Q_x^2\, \varphi_{zz}\, dx.}$$

Nonlocal/Nonlocal Formulation - Summary / Comments

missing ε

Integral Relation A

$$\int_{\mathscr{S}} \frac{d}{dt}\varphi \, dx = \int_{\mathscr{S}} \varepsilon\phi \left(\nabla\varphi_z \cdot \mathbf{n}\right) \, dx - \int_{\mathscr{B}} \underline{\varepsilon Q \varphi_{zz} \, dx}$$

$$\boxed{Q(x,t) = \phi(x,-\mu,t)}$$

Integral Relation B

$$\int_{\mathscr{S}} \frac{d}{dt}\epsilon q \left(\frac{\partial\varphi_z}{\partial\mathbf{n}} - 2\varphi_{zz}\right) \, dx = \int_{\mathscr{S}} \epsilon\mu\eta(\varphi_{zz} + \epsilon\mu\eta_x\varphi_{xz}) - \underbrace{2\epsilon^2\mu\eta_t \, q \, \varphi_{zzz}}_{(*)} \, dx + \int_{\mathscr{B}} \underline{\frac{\epsilon^2}{2} Q_x^2 \, \varphi_{zz} \, dx}$$

Nonlocal/Nonlocal Formulation - Summary / Comments

Integral Relation A

missing an ε here.

$$\int_{\mathscr{S}} \frac{d}{dt}\varphi \, dx = \int_{\mathscr{S}} \varepsilon \, \phi \left(\nabla \varphi_z \cdot \mathbf{n}\right) \, dx - \int_{\mathscr{B}} \varepsilon \, Q \varphi_{zz} \, dx$$

$$\boxed{Q(x,t) = \phi(x, -\mu, t)}$$

Integral Relation B

$$\int_{\mathscr{S}} \frac{d}{dt}\epsilon q \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz} \right) \, dx = \int_{\mathscr{S}} \epsilon \mu \eta (\varphi_{zz} + \epsilon \mu \eta_x \varphi_{xz}) - \underbrace{2\epsilon^2 \mu \eta_t \, q \, \varphi_{zzz}}_{(*)} \, dx + \int_{\mathscr{B}} \frac{\epsilon^2}{2} Q_x^2 \, \varphi_{zz} \, dx$$

Remark #1

$\varphi = e^{-ikx}\sinh(k(z+\mu))$, implies both **[ZCS]** and **[AFM]**.

If we take $k \to 0$ in both (A) and (B), we recover (T1) and (~~T5~~ T3) immediately.

Nonlocal/Nonlocal Formulation - Summary / Comments

missing an ε here.

Integral Relation A

$$\int_{\mathscr{S}} \frac{d}{dt}\varphi \, dx = \int_{\mathscr{S}} \varepsilon\, \phi \, (\nabla \varphi_z \cdot \mathbf{n}) \, dx - \int_{\mathscr{B}} \varepsilon\, \underset{Q_t}{Q} \varphi_{zz} \, dx$$

$$\boxed{Q(x,t) = \phi(x, -\mu, t)}$$

Integral Relation B

$$\int_{\mathscr{S}} \frac{d}{dt} \epsilon q \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz} \right) dx = \int_{\mathscr{S}} \epsilon\mu\eta(\varphi_{zz} + \epsilon\mu\eta_x \varphi_{xz}) - \underbrace{2\epsilon^2 \mu \eta_t \, q \, \varphi_{zzz}}_{(*)} \, dx + \int_{\mathscr{B}} \frac{\epsilon^2}{2} Q_x^2 \, \varphi_{zz} \, dx$$

Remark #2

We can generate a direct map from the **pressure at the bottom** to the free-surface variables. This can be found by taking the combination

$$\longrightarrow \qquad \frac{d}{dt}(A) - (B)$$

Nonlocal/Nonlocal Formulation - Summary / Comments

Integral Relation A

$$\int_{\mathscr{S}} \frac{d}{dt}\varphi \, dx = \int_{\mathscr{S}} \phi \left(\nabla\varphi_z \cdot \mathbf{n}\right) \, dx - \int_{\mathscr{B}} Q\varphi_{zz} \, dx \qquad \boxed{Q(x,t) = \phi(x,-\mu,t)}$$

Integral Relation B

$$\int_{\mathscr{S}} \frac{d}{dt}\epsilon q \left(\frac{\partial\varphi_z}{\partial\mathbf{n}} - 2\varphi_{zz}\right) \, dx = \int_{\mathscr{S}} \epsilon\mu\eta(\varphi_{zz} + \epsilon\mu\eta_x\varphi_{xz}) - \underbrace{2\epsilon^2\mu\eta_t \, q \, \varphi_{zzz}}_{(*)} \, dx + \int_{\mathscr{B}} \frac{\epsilon^2}{2} Q_x^2 \, \varphi_{zz} \, dx$$

Remark #3

There is a suggestive **Legendre-type Transform** lurking in (B).

Nonlocal/Nonlocal Formulation - Summary / Comments

Integral Relation A

$$\int_{\mathscr{S}} \frac{d}{dt}\varphi \, dx = \int_{\mathscr{S}} \phi \left(\nabla \varphi_z \cdot \mathbf{n} \right) \, dx - \int_{\mathscr{B}} Q \varphi_{zz} \, dx \qquad \boxed{Q(x,t) = \phi(x, -\mu, t)}$$

Integral Relation B

$$\int_{\mathscr{S}} \frac{d}{dt} \epsilon q \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz} \right) \, dx = \int_{\mathscr{S}} \epsilon\mu\eta(\varphi_{zz} + \epsilon\mu\eta_x \varphi_{xz}) - \underbrace{2\epsilon^2 \mu \eta_t \, q \, \varphi_{zzz}}_{(*)} \, dx + \int_{\mathscr{B}} \frac{\epsilon^2}{2} Q_x^2 \, \varphi_{zz} \, dx$$

Remark #3

There is a suggestive **Legendre-type Transform** lurking in (B).

This observation allows us to recover the Hamiltonian.

Nonlocal/Nonlocal Formulation - Summary / Comments

Integral Relation A

$$\int_{\mathscr{S}} \frac{d}{dt}\varphi \, dx = \int_{\mathscr{S}} \phi \left(\nabla \varphi_z \cdot \mathbf{n}\right) \, dx - \int_{\mathscr{B}} Q \varphi_{zz} \, dx$$

$$\boxed{Q(x,t) = \phi(x,-\mu,t)}$$

Integral Relation B

$\phi_t + \frac{1}{2}|\nabla\phi|^2 = f(\;)$

$$\int_{\mathscr{S}} \frac{d}{dt}\epsilon q \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz}\right) \, dx = \int_{\mathscr{S}} \left(\epsilon\mu\eta (\varphi_{zz} + \epsilon\mu\eta_x \varphi_{xz}) - \underbrace{2\epsilon^2 \mu \eta_t \, q \, \varphi_{zzz}}_{(*)} \right) dx + \int_{\mathscr{B}} \frac{\epsilon^2}{2} Q_x^2 \, \varphi_{zz} \, dx$$

Remark #4

The **dynamic boundary condition** was only prescribed at the last stage of deriving (B). This will return at a later stage.

Nonlocal/Nonlocal Formulation - Summary / Comments

Integral Relation A

$$\int_{\mathscr{S}} \frac{d}{dt}\varphi\, dx = \int_{\mathscr{S}} \phi\,(\nabla\varphi_z \cdot \mathbf{n})\; dx - \int_{\mathscr{B}} Q\varphi_{zz}\, dx$$

$$\boxed{Q(x,t) = \phi(x,-\mu,t)}$$

Integral Relation B

$$\int_{\mathscr{S}} \frac{d}{dt}\epsilon q\left(\frac{\partial\varphi_z}{\partial\mathbf{n}} - 2\varphi_{zz}\right)\; dx = \int_{\mathscr{S}} \epsilon\mu\eta(\varphi_{zz} + \epsilon\mu\eta_x\varphi_{xz}) - \underbrace{2\epsilon^2\mu\eta_t\, q\, \varphi_{zzz}}_{(*)}\, dx + \int_{\mathscr{B}} \frac{\epsilon^2}{2}Q_x^2\, \varphi_{zz}\, dx$$

Remark #5

Are there conservation laws **hiding in plain sight** here?

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A Different View of the Nonlocal/Nonlocal Formulation

Modifying our integral relations:

$$\boxed{q = \phi(x, \epsilon\mu\eta, t), \;\; Q = \phi(x, -\mu, t)}$$

Integral Relation A

$$\int_{\mathscr{S}} \frac{d}{dt}\left(\varphi - \epsilon t q \frac{\partial \varphi_z}{\partial \mathbf{n}}\right)\, dx = -\epsilon t \int_{\mathscr{S}} \frac{d}{dt}\left(q \frac{\partial \varphi_z}{\partial \mathbf{n}}\right)\, dx - \int_{\mathscr{B}} Q\, \varphi_{zz}\, dx$$

Integral Relation B

$$\int_{\mathscr{S}} \frac{d}{dt} \epsilon q \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz}\right)\, dx = \int_{\mathscr{S}} \epsilon\mu\eta(\varphi_{zz} + \epsilon\mu\eta_x\varphi_{xz}) - 2\epsilon^2\mu\eta_t\, q\, \varphi_{zzz}\, dx + \int_{\mathscr{B}} \frac{\epsilon^2}{2} Q_x^2\, \varphi_{zz}\, dx$$

A Different View of the Nonlocal/Nonlocal Formulation

Modifying our integral relations:

$$\boxed{q = \phi(x, \epsilon\mu\eta, t),\ \ Q = \phi(x, -\mu, t)}$$

Integral Relation A

$$\int_{\mathscr{S}} \frac{d}{dt}\left(\varphi - \epsilon t q \frac{\partial \varphi_z}{\partial \mathbf{n}}\right)\, dx = -\epsilon t \int_{\mathscr{S}} \frac{d}{dt}\left(q \frac{\partial \varphi_z}{\partial \mathbf{n}}\right)\, dx - \int_{\mathscr{B}} Q\, \varphi_{zz}\, dx$$

Integral Relation B

$$\int_{\mathscr{S}} \frac{d}{dt}\epsilon q\left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz}\right)\, dx = \int_{\mathscr{S}} \epsilon\mu\eta(\varphi_{zz} + \epsilon\mu\eta_x\varphi_{xz}) - 2\epsilon^2\mu\eta_t\, q\, \varphi_{zzz}\, dx + \int_{\mathscr{B}} \frac{\epsilon^2}{2} Q_x^2\, \varphi_{zz}\, dx$$

Remark #1

Notice that (A) and (B) have a term in common. This will be useful and will allow for a recursive process.

A Different View of the Nonlocal/Nonlocal Formulation

Modifying our integral relations:

$$\boxed{q = \phi(x, \epsilon\mu\eta, t), \;\; Q = \phi(x, -\mu, t)}$$

Integral Relation A

$$\int_{\mathscr{S}} \frac{d}{dt}\left(\varphi - \epsilon t q \frac{\partial \varphi_z}{\partial \mathbf{n}}\right) dx = -\epsilon t \int_{\mathscr{S}} \frac{d}{dt}\left(q \frac{\partial \varphi_z}{\partial \mathbf{n}}\right) dx - \int_{\mathscr{B}} Q\, \varphi_{zz}\, dx$$

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Remark #2

IBP introduced higher derivatives of φ. If those eventually disappear, we may be able to find our conservation laws. **Polynomials seem like a natural choice.**

Turning the Crank

Integral Relation A

$$\varphi = x + iz \quad \rightarrow \quad \int_{\mathscr{S}} \frac{d}{dt}\left(x + i\epsilon\mu\eta\right)\, dx = 0.$$

Turning the Crank

Integral Relation A

$$\varphi = x + iz \quad \to \quad \int_{\mathscr{S}} \frac{d}{dt}\left(x + i\epsilon\mu\eta\right)\, dx = 0.$$

Integral Relation B

$$\varphi = \frac{1}{2}(x+iz)^2 \to \frac{d}{dt}\int_{\mathscr{S}} q(1 - i\epsilon\mu\eta_x) = -\frac{d}{dt}\int_{\mathscr{S}} t\eta\, dx = -\int_{\mathscr{B}} \frac{\epsilon}{2} Q_x^2\, dx.$$

Turning the Crank

Integral Relation A

$$\varphi = x + iz \quad \rightarrow \quad \int_{\mathscr{S}} \frac{d}{dt} \left(x + i\epsilon\mu\eta \right) \, dx = 0.$$

Integral Relation B

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Real/Imaginary Parts

(T3) $$\frac{d}{dt}\int_{\mathscr{S}} \eta \, dx = 0$$

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Real/Imaginary Parts

(T3) $$\frac{d}{dt}\int_{\mathscr{S}} \eta\, dx = 0$$

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Turning the Crank

Integral Relation A

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Real/Imaginary Parts

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(T1) $$\frac{d}{dt}\int_{\mathscr{S}} -q\eta_x\, dx = 0$$

Turning the Crank

Integral Relation A

$$\varphi = \frac{1}{2}(x+iz)^2 \quad \to \quad \int_{\mathscr{S}} \frac{d}{dt}\frac{1}{2}(x+i\epsilon\mu\eta)^2 + \epsilon t q(1+i\epsilon\mu\eta_x)\,dx$$

$$= \underbrace{t\frac{d}{dt}\int_{\mathscr{S}} \epsilon q(1-i\epsilon\mu\eta_x)\,dx}_{*} + 2it\underbrace{\frac{d}{dt}\int_{\mathscr{S}} \epsilon\mu\eta_x q\,dx}_{**} + \int_{\mathscr{B}} \epsilon Q\,dx$$

Turning the Crank

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$$\varphi = \frac{1}{2}(x+iz)^2 \quad \rightarrow \quad \int_{\mathscr{S}} \frac{d}{dt}\frac{1}{2}(x+i\epsilon\mu\eta)^2 + \epsilon t q(1+i\epsilon\mu\eta_x)\,dx$$

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Note that we have already dealt with $(*)$ and $(**)$ and can back-substitute these relationships.

future note

It might be worthwhile to show how to collapse this explicitly.

Real/Imaginary Parts

(T6) $$\frac{d}{dt}\int_{\mathscr{S}} -\frac{\epsilon^2\mu^2}{2}\eta^2 + \epsilon t q + \frac{\epsilon t^2}{2}\eta\,dx = \int_{\mathscr{B}} \epsilon Q - t\frac{\epsilon^2}{2}Q_x^2\,dx$$

(T5) $$\frac{d}{dt}\int_{\mathscr{S}} x\eta + t\epsilon q\eta_x\,dx = 0$$

Turning the Crank

Integral Relation B with $\varphi = \frac{1}{6}(x+iz)^3$: the full expression is a bit complicated:

$$\frac{d}{dt}\int_{\mathscr{S}} q(1-i\epsilon\mu\eta_x)(x+i\epsilon\mu\eta)-t\left(4i\mathscr{H} - x\eta - \frac{7i\epsilon\mu}{2}\eta^2\right)-\frac{t^2}{2\mu}q(1-7i\epsilon\mu\eta_x)-\frac{t^3}{6\mu}\eta\,dx = \frac{t}{\mu}\int_{\mathscr{B}} -Q\,dx+\frac{t^2}{2\mu}\int_{\mathscr{B}}\frac{\epsilon}{2}Q_x^2\,dx-\int_{\mathscr{B}}\frac{\epsilon}{2}Q_x^2(x-i\mu)\,dx$$

where we have introduced the notation

$$\mathscr{H} = \frac{\epsilon\mu}{2}\left(q\eta_t + \mu\eta^2\right)$$

Turning the Crank

Integral Relation B with $\varphi = \frac{1}{6}(x+iz)^3$~~.~~ the full expression is a bit complicated:

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Real/Imaginary Parts

$$\text{(T8)}\qquad \frac{d}{dt}\int_{\mathscr{S}}\left[q(x+\epsilon^2\mu^2\eta\eta_x) - tx\eta - \frac{t^2}{2\mu}q - \frac{t^3}{6\mu}\eta\right]dx = \int_{\mathscr{B}}\left[\left(\frac{t^2}{2\mu}-x\right)\frac{\epsilon}{2}Q_x^2 - \frac{t}{\mu}Q\right]dx$$

$$\text{(T7)}\qquad \frac{d}{dt}\int_{\mathscr{S}}\left[\epsilon\mu q(\eta - x\eta_x) - t\left(4\mathscr{H} - \frac{7\epsilon\mu}{2}\eta^2\right) + \frac{7\epsilon\mu t^2}{2\mu}q\eta_x\right]dx = \int_{\mathscr{B}}\frac{\epsilon\mu}{2}Q_x^2\,dx$$

Summary thus far...

At this point, we have found the eight conservation laws found in **[Benjamin & Olver, Olver]**

	φ
$T_1 = -\eta_x \Phi$	Relation B $\psi = \frac{1}{2}(x+iz)^2$
$T_2 = \dfrac{1}{2}\Phi\Phi_{(n)} + \dfrac{1}{2}g\eta^2$	Relation A **Legendre Hint**
$T_3 = \eta$	Relation A $\psi = x + iz$
$T_4 = \Phi + gt\eta$	Relation B $\psi = \frac{1}{2}(x+iz)^2$
$T_5 = x\eta + t\eta_x\Phi$	Relation A $\psi = \frac{1}{2}(x+iz)^2$
$T_6 = \dfrac{1}{2}\eta^2 - t\Phi - \dfrac{1}{2}gt^2\eta$	Relation A $\psi = \frac{1}{2}(x+iz)^2$
$T_7 = (\eta - x\eta_x)\Phi - t(4T_2 - 7gT_6) + \dfrac{7}{2}gt^2T_4 - \dfrac{7}{6}g^2t^3T_3$	Relation B $\psi = \frac{1}{6}(x+iz)^3$
$T_8 = (x + \eta\eta_x)\Phi + gtT_5 + \dfrac{1}{2}gt^2T_1$	Relation B $\psi = \frac{1}{6}(x+iz)^3$

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It is worth noting that we were able to find (T7) directly.

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$T_1 = -\eta_x \Phi$	Relation B $\psi = \frac{1}{2}(x+iz)^2$ Should be φ
$T_2 = \frac{1}{2}\Phi\Phi_{(n)} + \frac{1}{2}g\eta^2$	Relation A **Legendre Hint**
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But what if we kept going? Note: we haven't used **Integral Relation A** with $\varphi = \frac{1}{6}(x+iz)^{\cancel{2}}$ 3

Some Remarks

Summary of **Integral Relation A** with $\varphi = \frac{1}{6}(x + iz)^3$

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- **Integral Relation A** with $\varphi = \frac{1}{3}(x+iz)^3$ will not fit on the slide. Though systematic, the back substitution become tedious.

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Summary of **Integral Relation A** with $\varphi = \frac{1}{6}(x+iz)^3$

- **Integral Relation A** with $\varphi = \frac{1}{3}(x+iz)^3$ will not fit on the slide. Though systematic, the back substitution become tedious.

- We have yet to find new non-trivial conservation laws in local coordinates. You can think of a density F as a **~~non~~-trivial conserved density** if

$$\frac{d}{dt}\int_{\mathscr{S}} F\,dx = 0 \text{ and } \int_{\mathscr{S}} F_{\cancel{,t}\,x}\,dx = 0.$$

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So far, we haven't been able to show that the remainder satisfies this condition fully. Still leaving some hope.

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Ok, so does this work for a 2D surface (3D Fluid)?

Extending to a Two Dimensional Surface...

At this point, we have found the 12 conservation laws found in **[Benjamin & Olver]**

$$T_1 = -\eta_x \Phi, \quad T_2 = -\eta_y \Phi \qquad T_3 = \frac{1}{2}\Phi\Phi_{(n)} + \frac{1}{2}g\eta^2$$

$$T_4 = \eta \qquad T_5 = \Phi + gt\eta$$

$$T_6 = x\eta + t\eta_x\Phi, \qquad T_7 = y\eta + t\eta_y\Phi$$

$$T_8 = \frac{1}{2}\eta^2 - t\Phi - \frac{1}{2}gt^2\eta \qquad T_9 = (x\eta_y - y\eta_x)\Phi$$

$$T_{10} = (x + \eta\eta_x)\Phi + gtT_6 + \tfrac{1}{2}gt^2T_1 \qquad T_{11} = (y + \eta\eta_y)\Phi + gtT_7 + \frac{1}{2}gt^2T_1$$

$$T_{12} = (\eta - x\eta_x - y\eta_y)\Phi + t(9gT_8 - 5\mathscr{H}) + \frac{9}{2}gt^2T_5 - \frac{3}{2}g^2t^3T_4$$

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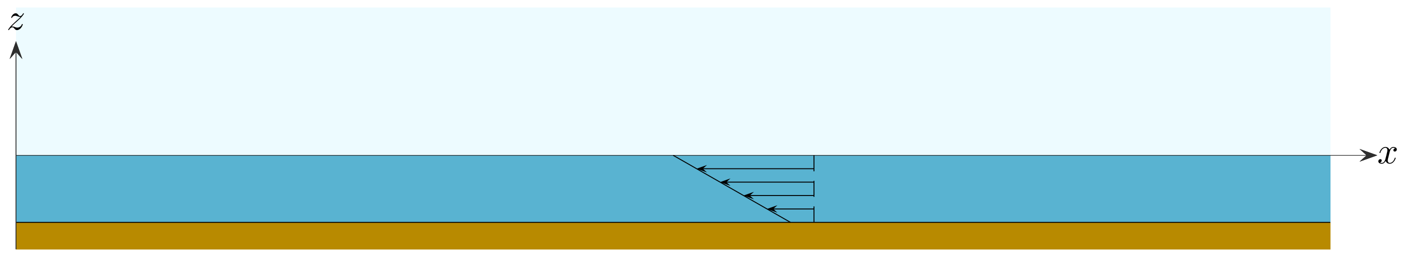
It is worth noting that T_9 and T_10 were found by taking the combination

This should be T_{12}

$$\varphi = \frac{1}{6}(y + iz)^3 - \frac{z}{2}(x + iy)^2.$$

Table of Contents

Constant Vorticity



wrong equations...

correct this slide.

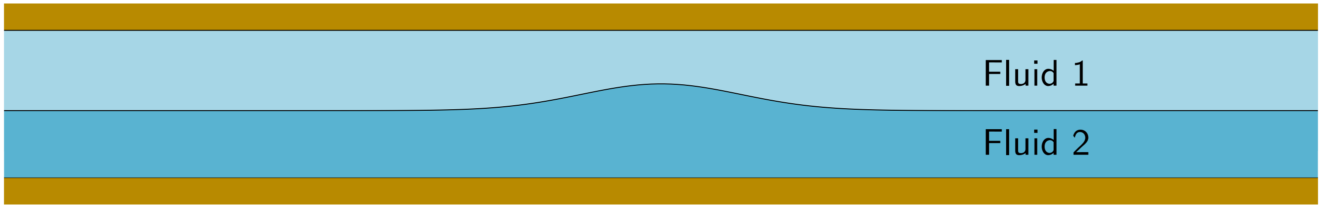
Integral Relation A

$$\frac{d}{dt}\int_{\mathscr{S}}(1-\rho)\varphi-\epsilon t\tilde{q}\,\frac{\partial\varphi_z}{\partial\mathbf{n}}\,dx=-\epsilon t\frac{d}{dt}\int_{\mathscr{S}}\tilde{q}\,\frac{\partial\varphi_z}{\partial\mathbf{n}}\,dx-\epsilon\int_{\mathscr{B}_1}Q_1\varphi_{zz}\,dx+\epsilon\rho\int_{\mathscr{S}}Q_2\varphi_{zz}\,dz$$

Integral Relation B

$$\frac{d}{dt}\int_{\mathscr{S}}\tilde{q}\left(\frac{\partial\varphi_z}{\partial\mathbf{n}}-2\varphi_{zz}\right)dx=\int_{\mathscr{S}}\left[-2\epsilon\mu\eta_t\tilde{q}\,\varphi_{zzz}+(1-\rho)\eta\left(\varphi_{zz}+\epsilon\mu\eta_x\varphi_{xz}\right)\right]dx$$
$$+\int_{\mathscr{B}_1}\frac{\epsilon}{2}Q_{1,x}^2\varphi_{zz}\,dx-\rho\int_{\mathscr{B}_2}\frac{\epsilon}{2}Q_{2,x}^2\varphi_{zz}\,dx$$

Rigid Lid



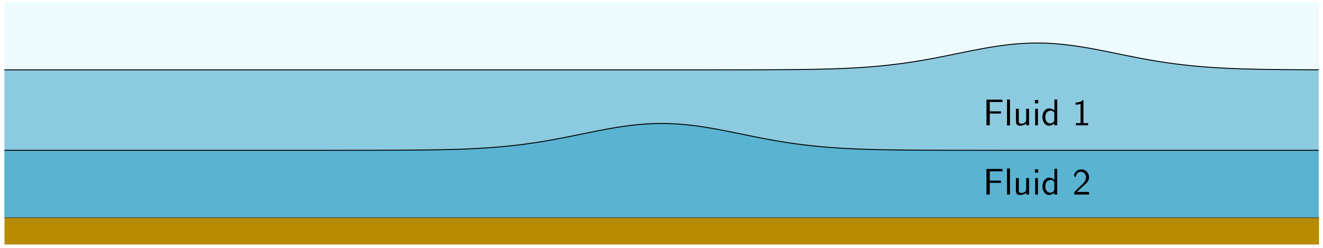
Integral Relation A

$$\frac{d}{dt}\int_{\mathscr{S}} (1-\rho)\varphi - \epsilon t \tilde{q}\,\frac{\partial \varphi_z}{\partial \mathbf{n}}\,dx = -\epsilon t\frac{d}{dt}\int_{\mathscr{S}} \tilde{q}\,\frac{\partial \varphi_z}{\partial \mathbf{n}}\,dx - \epsilon\int_{\mathscr{B}_1} Q_1\varphi_{zz}\,dx + \epsilon\rho\int_{\mathscr{S}} Q_2\varphi_{zz}\,dz$$

Integral Relation B

$$\begin{aligned}\frac{d}{dt}\int_{\mathscr{S}} \tilde{q}\left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz}\right)dx = \int_{\mathscr{S}} &\left[-2\epsilon\mu\eta_t\tilde{q}\,\varphi_{zzz} + (1-\rho)\eta\left(\varphi_{zz} + \epsilon\mu\eta_x\varphi_{xz}\right)\right]\,dx \\ &+ \int_{\mathscr{B}_1}\frac{\epsilon}{2}Q_{1,x}^2\varphi_{zz}\,dx - \rho\int_{\mathscr{B}_2}\frac{\epsilon}{2}Q_{2,x}^2\varphi_{zz}\,dx\end{aligned}$$

Two Layers



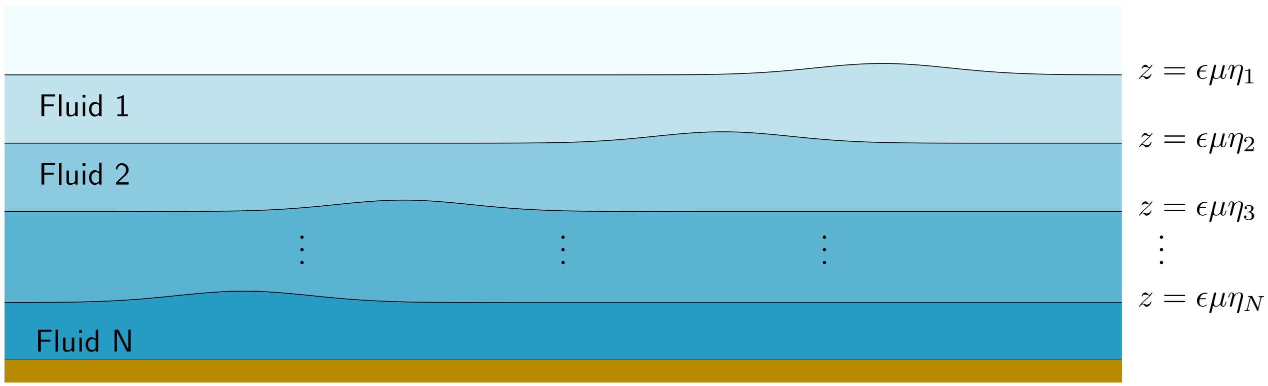
Integral Relation A

$$\frac{d}{dt}\left[\sum_j \int_{\mathscr{S}_j} \left[\tilde{\rho}_j \varphi - \epsilon t \tilde{q}_j \frac{\partial \varphi_z}{\partial \mathbf{n}}\right] dx\right] = -\epsilon t \frac{d}{dt}\left[\sum_j \int_{\mathscr{S}_j} \left[\tilde{q}_j \frac{\partial \varphi_z}{\partial \mathbf{n}}\right] dx\right]$$

Integral Relation B

$$\sum_j \frac{d}{dt} \int_{\mathscr{S}_j} \left[\tilde{q}_j \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz}\right)\right] dx = \sum_j \left[\int_{\mathscr{S}_j} \left[-2\epsilon\mu\eta_{j,t}\tilde{q}_j\varphi_{zzz} + (\mu\tilde{\rho}\eta_j)\left(\varphi_{zz} + \epsilon\mu\eta_{j,x}\varphi_{xz}\right)\right] dx\right]$$

N-Layers



Integral Relation A

$$\frac{d}{dt}\left[\sum_j \int_{\mathscr{S}_j} \left[\tilde{\rho}_j \varphi - \epsilon t \tilde{q}_j \frac{\partial \varphi_z}{\partial \mathbf{n}}\right] \, dx\right] = -\epsilon t \frac{d}{dt}\left[\sum_j \int_{\mathscr{S}_j} \left[\tilde{q}_j \frac{\partial \varphi_z}{\partial \mathbf{n}}\right] \, dx\right]$$

Integral Relation B

$$\sum_j \frac{d}{dt} \int_{\mathscr{S}_j} \left[\tilde{q}_j \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz}\right)\right] \, dx = \sum_j \left[\int_{\mathscr{S}_j} \left[-2\epsilon\mu\eta_{j,t}\tilde{q}_j\varphi_{zzz} + (\mu\tilde{\rho}\eta_j)\left(\varphi_{zz} + \epsilon\mu\eta_{j,x}\varphi_{xz}\right)\right] \, dx\right]$$

Additional Remarks/Summary

We have generated a **Nonlocal-Nonlocal formulation** and **derived conservation laws** for each of the following scenarios:

1. Irrotational fluid with one free surface (1D & 2D free surface) ✓
2. Constant Vorticity (1D free surface) near completion
3. Irrotational, rigid lid with two fluids of different densities (1D free surface)✓
4. Irrotational N-fluid layers with different densities (1D free surfaces)✓

We are currently working on

1. Formalizing the parametric representation interfaces (1D).
2. Classifying various dynamic boundary condition (surface tension, external forcing, etc.)
3. Considering specific external forcing at a side wall (wave-maker).
4. Repeating the results for constant vorticity in N-fluid layers with different densities (1D free surface).

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And with that, thank you for your attention & comments.